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# THE INFLUENCE OF ENTREPRENEURIAL EXPECTATIONS AND APPRAISALS ON PRODUCTION PLANNING: AN ECONOMETRIC STUDY OF THE GERMAN LEATHER AND SHOE INDUSTRIES*

BY T. THONSTAD AND D. B. JOCHEMS[1]

## 1. INTRODUCTION

DURING THE LAST ten years numerous articles discussing optimal production planning and optimal inventory policy have appeared.[2] Relatively little work has been done, however, on the econometric study of actual behavior with respect to production and inventories. This may, in part, be due to lack of data on some of the variables which appear very important from the theoretical point of view, particularly expectations regarding sales and prices and desired levels of stocks. This study attempts to analyze production planning in the German leather and shoe industry; it places particular stress on several kinds of expectations and "appraisals" as explanatory variables. We have been able to do this by virtue of the availability of Business Test data (supplied by the IFO Institut für Wirtschaftsforschung in Munich); these data, contrary to most conventional statistics, give information about *expectations* with respect to prices and demand and about entrepreneurial *appraisals* of stocks and order positions. For readers who may not be familiar with Business Tendency Surveys, we shall first briefly describe them; for further details we refer to Theil [14] and Marquardt and Strigel [13].

These surveys are based upon monthly questionnaires in which entrepreneurs are asked to state the *actual* direction of change of a certain

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[1] T. Thonstad's stay at the Econometric Institute of the Netherlands School of Economics was facilitated by grants from the Netherlands Ministry of Education and the Netherlands Universities Foundation for International Cooperation. This study was carried out under auspices of the Research Center of CIRET (Contact International de Recherches Economiques Tendancielles). The authors are indebted to Professors H. Theil and A. S. Goldberger for their valuable suggestions and helpful comments.

[2] For detailed references see Arrow, Karlin, and Scarf [4].

number of economic variables (e.g., production, stocks, orders received) in the present month, derived by comparing the level of the variable in that month with that of the previous month, as well as the *anticipated* direction of change in the next month. For realizations as well as for expectations and plans there are only three possible answers, viz., increase (+), no change (0), and decrease (−). The reports that are given on questions regarding entrepreneurial *appraisal* of the level of certain variables—like the appraisal of stocks of finished goods and the order position—have three possible answers, viz., "large or too large" (+), "normal" (0) and "small or too small" (−). The results of these surveys are disclosed to the participants[3] in the aggregative form of three weighted relative frequencies of entrepreneurs reporting increases, no change and decreases; weights are based on the relative sizes of the firms.

On the basis of these fractions of increase, no-change, and decrease, one can construct the so-called *balance*, which is defined as the fraction of increase minus the fraction of decrease. Such balances vary between plus one and minus one depending upon the extent to which "increase" or "decrease" is dominant; they are comparable (though, of course, not identical) with the relative changes in "ordinary" index numbers.[4] The interdependence of time series of such balances for price variables of the German leather and shoe industry was analyzed in great detail by Theil for the years 1951—53 [14]. The present study deals with the interrelationships between Business Test data on planned production and several other types of test data in the period 1956—58.

In Section 2 we consider some factors that are likely to influence production planning. The material used is described in Section 3, while in Section 4 regressions of planned production on single explanatory variables are presented. Multiple regressions, which estimate the influence of different factors simultaneously, are discussed in Section 5. In Section 6 we deal with the question of whether the "best" structure for 1956—58 still prevails in 1959.

## 2. SOME PRELIMINARY CONSIDERATIONS

In this section we shall discuss a number of possible partial expla-

[3] It is conceivable that the mere existence of the Business Test may have influenced the economic as well as the reporting behavior of the participating entrepreneurs, because each month they receive information about what has happened in the rest of their branch and sometimes in other branches as well; in particular, they receive information about the plans and expectations of their competitors. Virtually nothing is known about this possibility, but it should be kept in mind if one wants to generalize the results.

[4] For comparisons of this type see, e.g., Anderson, Bauer, and Giehl [3], Van Beeck, Engberts, and Lisman [5], and Jochems and De Wit [9, 10].

nations of the formation of entrepreneurs' production plans. If one accepts certain theoretical concepts and considers entrepreneurs as "quantity adapters," only actual and expected prices should be relevant as explanatory variables for planned production. In that case, expected demand, for example, should be quite irrelevant, since entrepreneurs by assumption think they can always sell as much as they want at the prevailing prices. But it is unlikely that this is true, particularly with respect to the industries considered in this study. It is more plausible to assume that entrepreneurs behave as if a demand curve is directed toward each of them, so that they will *plan* price changes (alone or together with some competitors) and react to present demand and to guesses about future demand. We have, therefore, deleted the price variables as explanatory variables of planned production.[5]

Perhaps the simplest assumption that can be made is that the planned changes in production are simply some kind of extrapolation of changes realized in the past.[6] For instance, the planned changes reported may be extrapolations of the actual change in production in the last month, or they may be extrapolations on the basis of the production change in the corresponding month a year ago.

It is often assumed that production depends upon demand with some lag. It therefore seems important to investigate empirically the possibility that the reactions of entrepreneurs to demand changes are lagged. Further, it is conceivable that they react primarily to changes in their *stock* of orders (or to the *appraisal* of this stock) rather than to changes in current demand. Lasting discrepancies between production and demand may lead to very large or very small stocks of finished goods. A usual assumption concerning the mechanism that restores "equilibrium" is that firms react to the deviation between actual and desired levels of stocks. Our Business Test variable "appraisal of the stock of finished goods" may tell something about this deviation. Finally, besides taking account of realizations in demand and inventory variables, the available data also give us an opportunity to investigate the role of expectations with respect to changes in demand and changes in "general business conditions."

It is conceivable that, within a given industry, the lag between

[5] It is worth mentioning here that Anderson [1, 2] has studied the reaction of entrepreneurs to unexpected changes in demand, as far as their prices or their production are concerned; apparently both variables are supposed to be entrepreneurial instruments.

[6] It should be emphasized that here and in the remainder of this paper the terms "production planning" and "planned production" mean "planned changes in production." Similarly, when we write "production," "stocks," "expected demand," etc., this should be interpreted as meaning "change in that variable."

order and delivery is reduced when general business conditions deteriorate, and that an industry may finally even change from producing for orders to producing for stock-building. This should imply that in certain periods essentially only order variables are important, and in other periods essentially only inventory variables matter. In this analysis we have not investigated this problem thoroughly, but have only tried to introduce both types of variables simultaneously in our equations. The *combined influence* on production planning of all the different factors discussed in this section will be treated in Section 5.

## 3. DESCRIPTION OF THE MATERIAL

Our analysis is based upon Business Test data for the period 1956—58, which were supplied by approximately seventy firms in the leather and seventy in the shoe industry. From 1956 on, considerably more data on expectations and appraisals have been collected than previously. Furthermore, it appears that the definitions of the concepts on the questionnaires are (with an unimportant exception) identical during this period but differed in some cases substantially from those of prior years. The same is true of the data for 1959 used for forecasting

TABLE 1

AVAILABILITY OF TEST DATA FOR 1956–58

| Variables | Realizations | Expectations and Plans | Appraisals |
|---|---|---|---|
| Production | $P$ | $\bar{P}$ | |
| Demand (new orders received) | $D$ | $\bar{D}$ | |
| Stock of orders | $O$ | | $O_A$ |
| Sales (deliveries)[a] | $S$ | | |
| Stocks of finished goods[b] | $F$ | | $F_A$ |
| General business conditions[c] | | $\bar{G}$ | • |
| Selling prices | • | • | |
| Export situation | • | | |
| Cost of materials | • | | |
| Stocks of raw materials | | | • |

[a] Sales in month $t$ compared with sales in the corresponding month last year ($t - 12$).

[b] In addition to the reports "increase" ($x^1$), "no change" ($x^2$), and "decrease" ($x^3$) there is also a fourth possibility for the variable "stocks of finished goods," viz., "no stocks" ($x^4$). We therefore have adjusted the usual balance series by taking into account changes in the fraction of firms reporting "no stocks", in the sense that when in successive months $(x_t^4 - x_{t-1}^4) > 0$, the difference is added to $x_t^3$ (decrease in stocks) and $x_{t-1}^4$ to $x_t^2$ (no change). When $(x_t^4 - x_{t-1}^4) \leq 0$, $x_t^4$ is considered an addition to $x_t^3$.

[c] Expectations for the next half year: $t + 1$ through $t + 6$, for short $t + H$. This leads to the symbol $\bar{G}_{t+H}$.

purposes in Section 6.

A survey of the variables for which B. T. data covering 1956—58 are available is presented in Table 1. The variables that finally are considered in this study are indicated by their corresponding symbols. Those variables that have been deleted for various reasons are indicated by a dot (·). It is seen, in Table 1, that the data on realizations and expectations—the latter denoted by a bar over the symbol—refer, with few exceptions, to changes in successive monthly levels. In the following sections we use subscripts to indicate the time period, e.g.,

$D_t$ = change in actual demand from month $t-1$ to month $t$;
$\bar{D}_{t+1}$ = expected change in demand from month $t$ to month $t+1$;
$(O_A)_t$ = appraisal of order position at the end of month $t$.

## 4. THE SINGLE-VARIABLE APPROACH

**4.1.** *Introduction.* Our general approach will be to describe the balance[7] of planned production linearly in terms of the balances of various explanatory variables:

$$b(\bar{P})_{t+1} = \alpha + \beta_1 b(x_1)_t + \beta_2 b(x_2)_t + \cdots + u_{t+1}, \tag{1}$$

where $b(\bar{P})_{t+1}$ is the balance of planned production for month $t+1$, $b(x_1)_t$, $b(x_2)_t$, $\cdots$ are balances of explanatory variables, $\alpha$, $\beta_1$, $\beta_2$, $\cdots$ are the parameters to be estimated, and $u_{t+1}$ is a random disturbance. Since all our variables are of the balance type, we shall simplify the notation by deleting the $b$-symbol and the brackets. Hence, the balance of planned production for month $t+1$ will be simply written as $\bar{P}_{t+1}$ and will be referred to as "planned production."

The estimation procedure used is least-squares; we shall first consider a number of regressions on single variables. They should be regarded mainly as background for multiple regression analyses, since it will be obvious that—given the variety of possible explanations of planned production—a single-cause theory has little chance of being preferable to more complex theories. In only a few cases, such as that of the appraisal variables discussed in 4.4, do these regressions contain explanatory variables that can be considered as meaningful single-variable explanations of planned changes in production. Table 2 presents thirty regressions arranged in four groups; they will be referred to as 1a, 1b, $\cdots$, 15a, 15b, where *a* refers to the tanners and *b* to the shoe manufacturers. If we take line 1 as our example, the table should be read in the following manner:

[7] A balance is defined as $b(x) = x^1 - x^2$, where $x^1$ and $x^2$ are weighted fractions of entrepreneurs reporting increases and decreases respectively; cf. Section 1.

| Tanners | Shoe Manufacturers |
|---|---|
| $\bar{P}_{t+1} = \underset{(0.01)}{0.03} + \underset{(0.07)}{0.33}\, P_t$ | $\bar{P}_{t+1} = \underset{(0.02)}{0.07} + \underset{(0.11)}{0.46}\, P_t$ |
| $\bar{R}^2 = 0.367$ | $\bar{R}^2 = 0.311$ |

The figures in parentheses are the estimated standard errors of the respective coefficients.

4.2. *Extrapolation variables.* Extrapolation from the change in actual production of the previous month, i.e., the regression of the balance of planned production for month $t+1$, $\bar{P}_{t+1}$, on the balance of actual production of month $t$, $P_t$, yields regression coefficients with positive signs and correlations of the order of 0.6.[8] This holds for both

TABLE 2

REGRESSIONS ON SINGLE VARIABLES

| Variables | Tanners | | | Shoe Manufacturers | | |
|---|---|---|---|---|---|---|
| | $\bar{R}^2$ | Constant Term | Regression Coefficient | $\bar{R}^2$ | Constant Term | Regression Coefficient |
| | Extrapolation Variables | | | | | |
| (1) $P_t$ | .367 | .03(.01) | .33(.07) | .311 | .07(.02) | .46(.11) |
| (2) $P_{t-11}$ | −.006 | .05(.02) | −.15(.11) | −.050 | .10(.04) | −.09(.17) |
| (3) $\bar{P}_t$ | −.055 | .04(.02) | .06(.17) | .551 | .02(.02) | .76(.11) |
| | Demand and Inventory Variables | | | | | |
| (4) $D_t$ | .093 | .04(.02) | .16(.07) | .188 | .09(.02) | .22(.06) |
| (5) $D_{t-1}$ | .250 | .04(.02) | .24(.06) | .355 | .08(.02) | .28(.06) |
| (6) $D_{t-2}$ | .017 | .04(.02) | .12(.07) | .235 | .08(.02) | .23(.06) |
| (7) $O_t$ | .036 | .05(.02) | .14(.08) | .187 | .09(.02) | .19(.06) |
| (8) $S_t$ | .227 | .05(.02) | .15(.04) | .401 | .04(.02) | .35(.07) |
| (9) $F_t$ | .276 | .04(.02) | −.21(.05) | .084 | .07(.02) | −.27(.12) |
| | Appraisal Variables | | | | | |
| (10) $(O_A)_t$ | .237 | .06(.02) | .22(.06) | .675 | .15(.02) | .47(.05) |
| (11) $(O_A)_{t-1}$ | .259 | .07(.02) | .23(.06) | .632 | .15(.02) | .46(.05) |
| (12) $(F_A)_t$ | .168 | .07(.02) | −.18(.06) | .402 | .13(.02) | −.49(.09) |
| (13) $(F_A)_{t-1}$ | .027 | .06(.02) | −.11(.07) | .348 | .12(.02) | −.46(.10) |
| | Expectation Variables | | | | | |
| (14) $\bar{D}_{t+1}$ | .082 | .01(.02) | .27(.12) | −.039 | .08(.03) | .09(.11) |
| (15) $\bar{G}_{t+H}$ | .019 | .02(.02) | .19(.11) | .415 | .06(.02) | .52(.10) |

[8] In the tables the squared correlation coefficients, $\bar{R}^2$, are given, whereas in the text we use the coefficient itself. The bar denotes here that the squared correlation coefficient

the tanners and the shoe manufacturers (regressions 1a and 1b). If we extrapolate from the change in actual production of the same month in the preceding year ($P_{t-11}$), no positive correlation is found. Extrapolation from planned production in the preceding month ($\bar{P}_t$) gives coefficients of positive sign and a correlation of the order of 0.75 for the shoe industry only (3b).

4.3. *Demand and inventory variables.* If we take changes in demand (new orders received, $D$), the correlations are not particularly high; for both tanners and shoe manufacturers the correlation reaches its maximum (0.5—0.6) when $D$ is taken with a lag of one month (5a and 5b). The coefficients and the regressions on "sales" (deliveries, $S$) and on "order position" (stock of orders, $O$) all have the required sign. The variable "stocks of finished goods," $F$, is an exceptional case (9a and 9b), since we obtain a better fit for the tanners than for the shoe manufacturers, contrary to almost all other regressions. For both, however, we observe the expected negative sign of the regression coefficient.

4.4. *Appraisal variables.* The regression on the variable "appraisal of the order position," $O_A$, yields a correlation of about 0.8, which is larger than all the others discussed so far. This applies only to the shoe manufacturers (10b); for the tanners, the correlations are around 0.5. It is conceivable that when entrepreneurs appraise their order position, they take into account not only their orders, but also their position with regard to stocks, and perhaps, to some extent, their expectations.[9] If this is true, we are justified in considering the single-variable regressions on $(O_A)_t$ as an acceptable explanation of production planning. Similarly, one can assume that when entrepreneurs appraise their stocks, $F_A$, they have already taken into account their order position, their expectations, etc. One may also interpret $F_A$ as an indication of the difference between desired and actual levels of stocks (as was done in Section 2). The regressions on $(F_A)_t$ are, however, less successful than those on $(O_A)_t$, so that these relations cannot be considered satisfactory single-variable explanations. For the tanneries, the correlation is low (about 0.4), but for the shoe manufacturers, the

---

has been corrected for loss of degrees of freedom, viz., $(1 - \bar{R}^2) = T(1 - R^2)/(T - A)$, where $T$ stands for the number of observations and $A$ for the number of parameters estimated. A similar correction has been applied to the standard errors of the regression coefficients.

[9] We make the obvious assumption that order appraisal (and similarly stock appraisal) and planned production are not simultaneously determined; entrepreneurs first make their appraisal, without taking into account their production plans, and then plan their production.

appraisal of the stock level of finished goods is somewhat more promising; both regressions show the expected negative sign.

4.5. *Expectation variables.* Finally, two expectation variables have been introduced, viz., expected demand and expected general business conditions. The first expectation refers to the same period as that of planned production, the second to the period of the next half year. For the shoe manufacturers, the variable "expected general business conditions" is the most successful in these single-variable regressions; for the tanners, none of the expectation variables is very successful.

4.6. *Constant terms.* It is worth noting in Table 2 that the constant term is consistently positive in almost all the regressions; in more than fifty per cent of the cases it is significantly different from zero. In view of the first difference character of most Business Test data, such constant terms stand for autonomous time trends and should not be very large. One possible explanation may be that linear regressions were fitted to non-linear relationships. For instance, let us assume that all entrepreneurs who experienced an actual increase in current month demand, $D_t$, plan to increase production accordingly, but that the reaction to decreases in demand is less uniform. In the latter case, some plan to decrease production, while others react differently. It can be shown that under these assumptions one may find a positive constant term.[10]

The extremely large constant terms in the regressions on appraisal variables cannot be entirely explained by this possible asymmetry in production planning. They seem to indicate that even when the majority of entrepreneurs appraises the order position, $x$, as unfavorable, $x^1 < x^2$, more increases than decreases in production, $y$, are planned, $y^1 > y^2$. It is conceivable that this phenomenon is due to some kind of "complaining bias"; entrepreneurs tend to appraise their order position pessimistically. Under this assumption, the balance, $b(x) = x^1 - x^2$, takes smaller values than the "true values"; it can be shown that this leads to a large positive constant term in the balance regression of $\bar{P}_{t+1}$ on $(O_t)$. For similar reasons, a positive constant term is to be expected in the regressions on stock appraisal.

## 5. THE MULTI-VARIABLE APPROACH

5.1. *Introduction.* Before proceeding to the multiple regressions

[10] Similarly, it is shown by Kloek and Jochems [11] that in the case of asymmetry found by Theil in his study of the price structure of the German leather and shoe branch in the period 1951–53 [14] a negative constant term is to be expected in the balance relation. His regressions of selling prices on buying prices for different groups of traders in this branch showed such negative constant terms.

which combine the single cause hypotheses of Section 2, we should take more explicit account of two fundamentally different patterns of production, namely, production to satisfy orders that have been received and production for stock building purposes, for future demand. In many cases firms devote their efforts to both goals at the same time, but the emphasis may vary from period to period according to the general business climate. In our case we may reasonably assume that the tanners are more engaged in production for stock building than are the shoe manufacturers, since their period of production is so much longer.[11] Consequently, we should expect the tanners to be more subject to inertia in their production planning, and this is undoubtedly one of the causes for the lower correlations of their production planning equations. Another important aspect is the fact that leather is more of a staple good than shoes which are a very heterogeneous, "fashion-influenced" product.[12]

5.2. *Results for the tanners.* The single-variable regressions of Table 2 served mainly to indicate a useful direction for further research. This led to a very large number of multiple-regression analyses for both industries. Since a complete account of all the results would be excessively lengthy and not necessarily enlightening, we shall confine ourselves to the following observations.

From Table 2 we see that $P_t$ is the "best" variable for the tanners as far as the correlation is considered (regression 1a); it is used in most of the multiple regressions. Other explanatory variables are expected demand for next month, $\bar{D}_{t+1}$, actual demand in months $t$ and $t-1$, stocks in month $t$, the appraisal of these stocks, and the appraisal of the order position. Still additional factors may influence the "true" relationship, but the following three alternatives seem to deserve further attention:

$$(2)\quad \bar{P}_{t+1} = \underset{(0.07)}{0.24}\, P_t + \underset{(0.08)}{0.34}\, \bar{D}_{t+1} + \underset{(0.06)}{0.15}\, D_{t-1} + \underset{(0.02)}{0.00}\,;$$

$$(3)\quad \bar{P}_{t+1} = \underset{(0.07)}{0.26}\, P_t + \underset{(0.08)}{0.28}\, \bar{D}_{t+1} + \underset{(0.06)}{0.15}\, D_{t-1} + \underset{(0.08)}{0.14}\, \bar{G}_{t+\pi} - \underset{(0.01)}{0.01}\,;$$

$$(4)\quad \bar{P}_{t+1} = \underset{(0.04)}{-0.25}\, F_t + \underset{(0.09)}{0.24}\, \bar{D}_{t+1} + \underset{(0.09)}{0.19}\, \bar{G}_{t+\pi} + \underset{(0.02)}{0.00}\,.$$

[11] The extent to which entrepreneurs report that they have no stocks differs also for the two industries: for the shoe manufacturers it is nine per cent on the average, for the tanners two per cent.

[12] This may also imply that shoe manufacturers do not always reduce their production when aggregate stocks are considered too large. For instance, if stocks of winter shoes are very large in the spring, manufacturers will not cut down their production of summer shoes.

The squared multiple correlations (corrected for loss of degrees of freedom) of these three regressions are: 0.604, 0.631, 0.522, respectively. The first two equations contain an extrapolation effect ($P_t$) and two demand variables. One of these is expected demand for next month,[13] the other, actual demand lagged one month. It may seem somewhat surprising that $D_{t-1}$ is used instead of $D_t$, but when $D_{t-1}$ is replaced by $D_t$, the resulting coefficient is not significant. It is not implausible, moreover, that a change in actual demand affects planned production only after some lag. Further, the second equation includes as a variable the expected general business conditions for the next half year; its coefficient is barely significant, but the introduction of this variable does raise the multiple correlation in spite of the loss of one degree of freedom. The third equation, (4), differs considerably from the first two in that $P_t$ is replaced by $F_t$ (stocks), with the expected negative sign. Both variables have their merits for the explanation of planned production.

In Figures 1—3 we have drawn "staple-diagrams" of our preferred relations over the period 1956-59. These diagrams—sometimes also referred to as Tinbergen-diagrams—show, for the period 1956-58, the contribution of the different explanatory variables to the explanation of $\bar{P}_{t+1}$, which resulted in the computed series $\bar{P}'_{t+1}$. The results for 1959, from which the series $\bar{P}^*_{t+1}$ was predicted, will be discussed in Section 6. It is seen that for 1956-58 the series of residuals of the regressions (2) and (3), which include $P_t$, show in some months fluctuations quite different from those of regression (4), which includes $F_t$. In order to test for autocorrelation in these series we have computed the von Neumann ratios[14] of the estimated disturbances. They have values of 2.61, 2.83, and 2.61 respectively, all exceeding the mean value of this ratio under the null hypothesis of independence; negative autocorrelation is thus indicated.[15] According to the tables given by Durbin and Watson [6] the ratio of the squared successive

[13] This variable contributes significantly to the explanation of $\bar{P}_{t+1}$ in almost all the regressions for the tanners. Some of these relations are not published in this paper.

[14] Cf. Tintner [17] and Durbin and Watson [6].

[15] It is seen in the diagrams that the negative balances of $\bar{P}_{t+1}$ and $P_t$ for September, 1957 are rather out of line in the series of otherwise positive balances in 1957. Further, the relatively large positive balance for October, 1957 seems to compensate the preceding value. It is possible that the large resulting residuals of opposite sign for both months are responsible for the sizable negative autocorrelation. An investigation by the IFO Institut in the reports of individual firms revealed that the negative balance values for $\bar{P}_{t+1}$ and $P_t$ are due to the fact that the largest—and hence heaviest weighted—tannery reported an actual and expected decline in production because of the actual and expected absenteeism caused by an epidemic of Asian influenza!

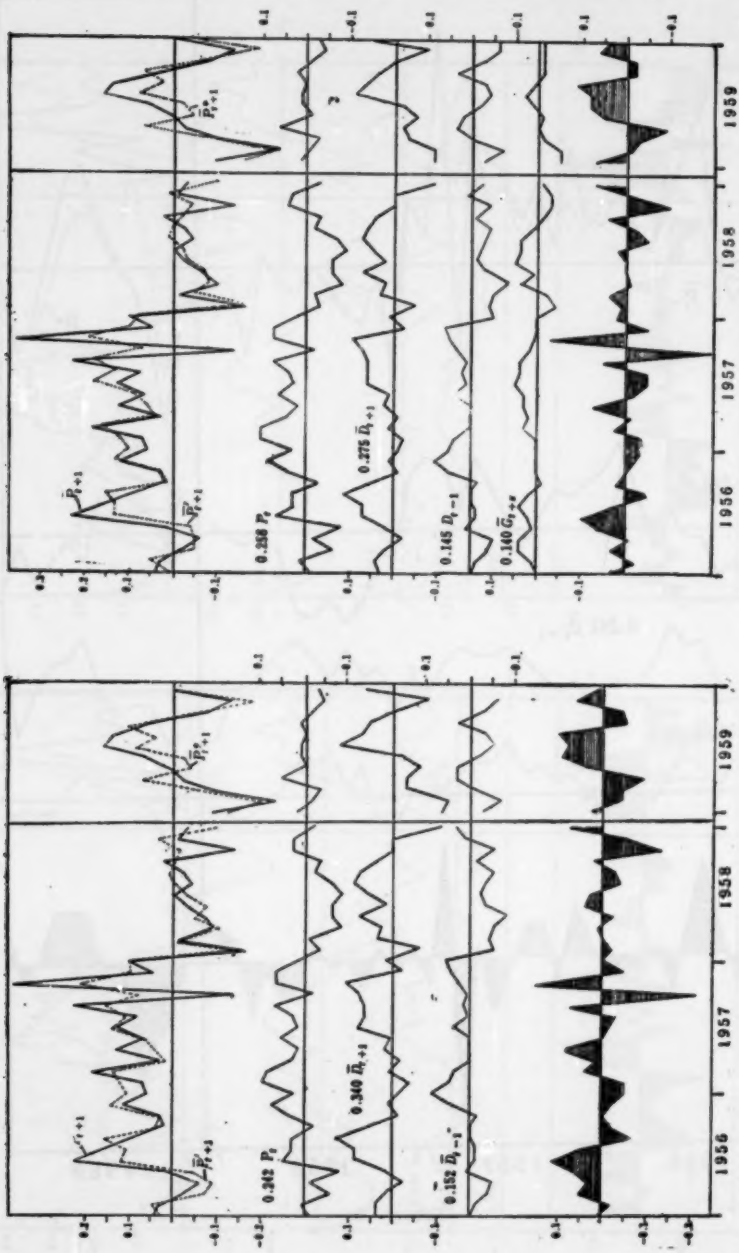


FIGURE 1




FIGURE 2

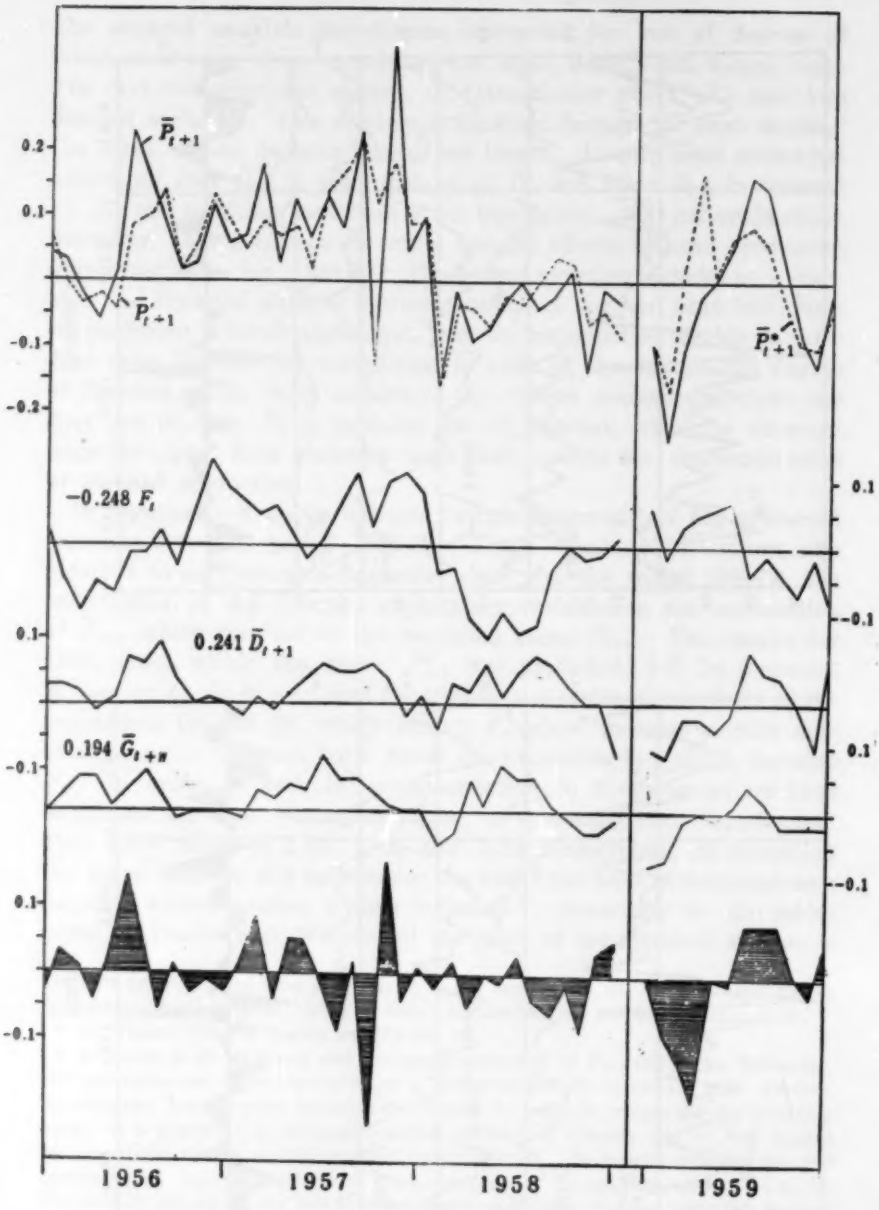
$\bar{P}_{t+1}$
$\bar{P}'_{t+1}$
$\bar{P}^{\bullet}_{t+1}$
0.2
0.1
-0.1
-0.2
$-0.248\ F_t$
$0.241\ \bar{D}_{t+1}$
$0.194\ \bar{G}_{t+n}$
0.1
-0.1
1956
1957
1958
1959

FIGURE 3

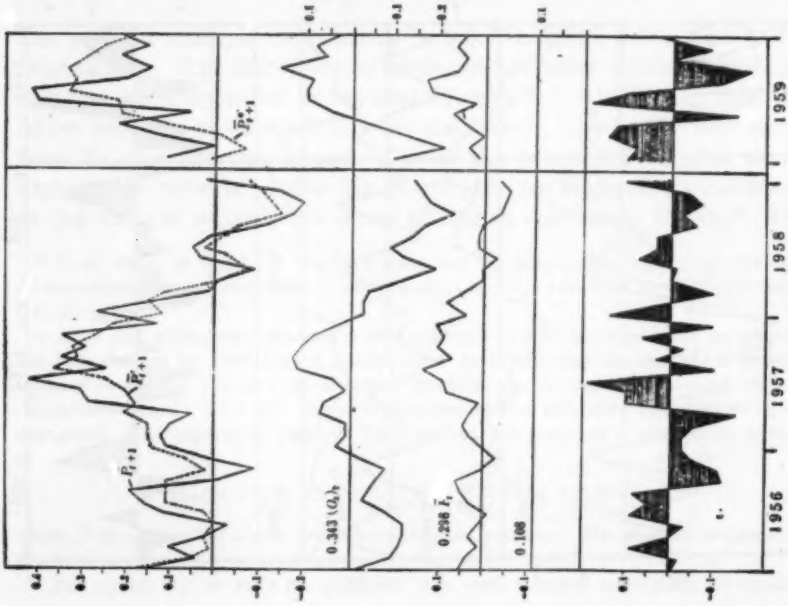


FIGURE 4

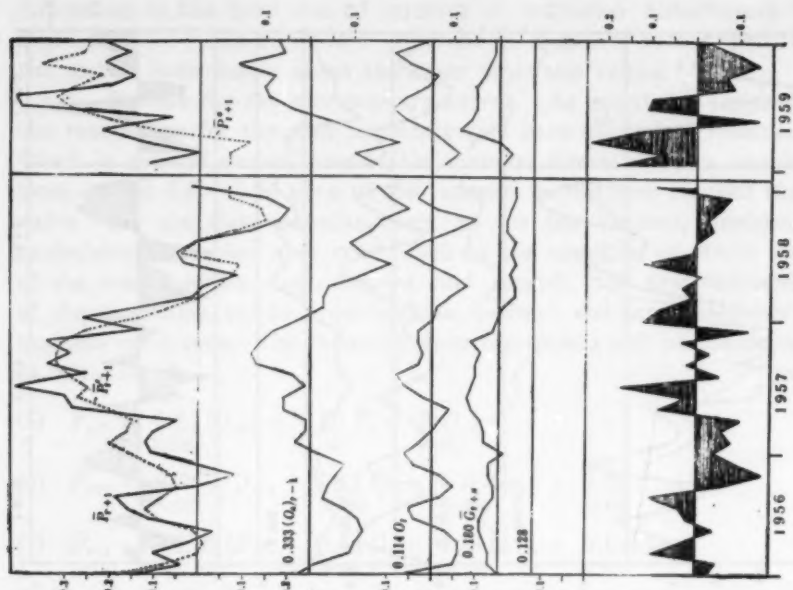


FIGURE 5

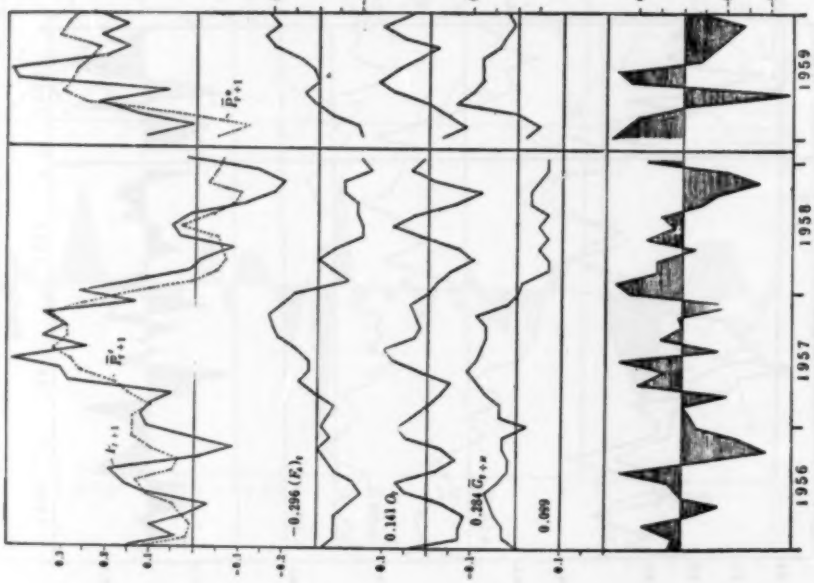


FIGURE 6

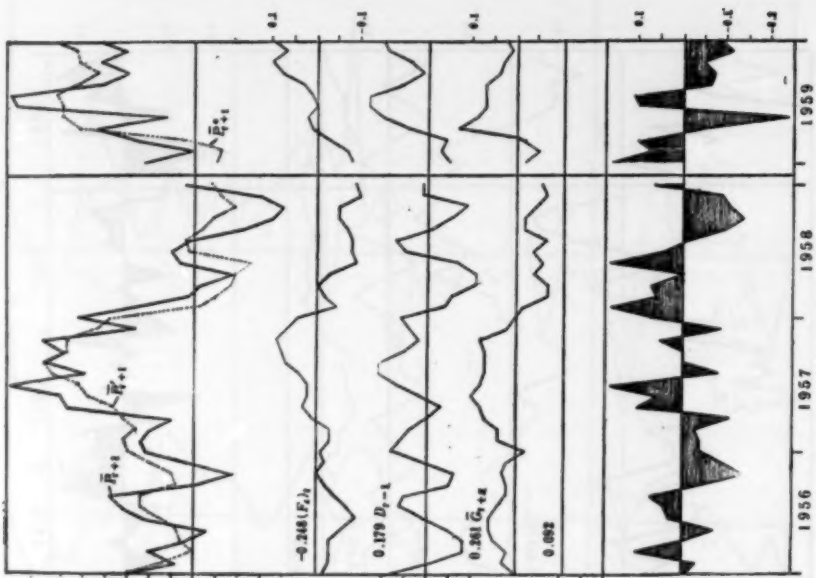


FIGURE 7

differences to the total sum of squares of estimated disturbances in regression (3) yields significant evidence of negative autocorrelation; the test is inconclusive about the other two ratio values.[16]

5.3. *Results for the shoe manufacturers.* As might be expected, the regressions for the shoe manufacturers have multiple correlations which in general exceed those of the tanners; several of these correlations exceed 0.8, while none of the tanners' correlations reached that value. For the shoe manufacturers, as for the tanners, numerous explanatory variables were tried, such as the appraisal of stocks and of the order position (both current and lagged), the first difference of these variable, stocks, order position, demand, and expected general business conditions. The following four regressions will be considered in more detail:

$$(5)\quad \bar{P}_{t+1} = \underset{(0.08)}{0.34}\,(O_A)_t + \underset{(0.14)}{0.30}\,\bar{P}_t + \underset{(0.03)}{0.11}\,;$$

$$(6)\quad \bar{P}_{t+1} = \underset{(0.07)}{0.33}\,(O_A)_{t-1} + \underset{(0.03)}{0.11}\,O_t + \underset{(0.09)}{0.18}\,\bar{G}_{t+II} + \underset{(0.02)}{0.12}\,;$$

$$(7)\quad \bar{P}_{t+1} = \underset{(0.09)}{-0.30}\,(F_A)_t + \underset{(0.04)}{0.14}\,O_t + \underset{(0.10)}{0.28}\,\bar{G}_{t+II} + \underset{(0.02)}{0.10}\,;^{17}$$

$$(8)\quad \bar{P}_{t+1} = \underset{(0.09)}{-0.25}\,(F_A)_t + \underset{(0.05)}{0.18}\,D_{t-1} + \underset{(0.09)}{0.26}\,\bar{G}_{t+II} + \underset{(0.02)}{0.09}\,.$$

The squared multiple correlations are 0.706, 0.736, 0.641, and 0.662, respectively. The first two equations contain order appraisal and the last two stock appraisal as explanatory variables. It is seen that the latter have lower correlations than the former. As to the first equation, (5), this has the lagged value of the dependent variable as an explanatory variable. Following Koyck [12], we interpret its coefficient as the ratio of a geometric series of lagged coefficients for $O_A$.[18] The

[16] In all cases, in which the Durbin-Watson test is inconclusive, significant evidence of autocorrelation is found when a test procedure recently developed by Theil and Nagar [16] is applied.

[17] After this study was completed a new approach to multiple regressions in balances has been devised by Theil [15]; it comes down to substituting the constant term by a variable containing the sum of weighted fractions of increase and decrease in the dependent variable $(y^1 + y^2)$. Some implications of this procedure have still to be investigated. For illustrative purposes Theil applied this approach to relation (7), leading to

$$(7')\quad \bar{P}_{t+1} = \underset{(0.09)}{-0.24}\,(F_A)_t + \underset{(0.04)}{0.12}\,O_t + \underset{(0.10)}{0.25}\,\bar{G}_{t+II} + \underset{(0.07)}{0.38}\,T_{t+1}$$

where $T$ stands for the above described additional variable. The residual variance of (7') is more than twice as small as the residual variance of (7).

[18] For current $O_A$, we have the coefficient 0.34 with standard error 0.08; for $(O_A)_{t-1}$,

second equation, (6), contains order appraisal lagged one month and the current actual order position. It is assumed that an entrepreneur applies his ideas about the size of his stock of orders after some lag as far as planned production is concerned and that he does so in the light of the change in the order position which occurs in the meantime.[19] This equation, as well as the two following, contains a variable on expected general business conditions for the next half year, its significance being slightly better than in the case of the tanneries. It is seen that in all four equations the large constant term (and hence the complaining bias discussed in Section 4) is preserved. We also find the expected negative signs for stock appraisal. A glance at Figures 4-7—where the series of $\bar{P}'_{t+1}$ values and the components of each relation are compared with the series of $\bar{P}_{t+1}$—tells us that the residuals of relation (5) show fluctuations that are somewhat different from those of the other series of residuals. The von Neumann ratios of these estimated disturbances amount to 2.04, 1.84, 1.55, and 1.44; the latter three suggest positive autocorrelation. There is no significant evidence of autocorrelation for the disturbances of regressions (5) and (6). For regressions (7) and (8) the test is inconclusive.[20]

5.4. *Concluding remarks.* It is usually more difficult to find high correlations for change than for absolute variables. Since most of the Business Test data are of the first-difference type, it is not surprising that we obtained relatively low correlations. There may also be some kind of irregularity in the reporting behavior of the entrepreneurs that in some sense corresponds to the effects of observational errors. Taking balances and disregarding the complete distribution of ( + ), (0), and ( − ) reports may also introduce errors. The asymmetric reaction pattern in production planning and the "complaining bias"—both discussed in 4.6—may be held responsible for the "unexpected" significant constant terms in regressions on single variables. In multiple regressions only some effects of the latter were found. The asymmetry may also have affected the correlations, since we fitted linear regressions to possibly non-linear relations. However, Theil obtained

$0.34 \times 0.30 = 0.10$ with asymptotic standard error 0.03; for $(O_A)_{t-2}$, $0.34 \times 0.30^2 = 0.03$ with asymptotic standard error 0.02; etc. The asymptotic standard errors are estimated from the formulas

$$\operatorname{var} r^n b = r^{2n} \operatorname{var} b + n^2 b^2 r^{2n-2} \operatorname{var} r + 2nbr^{2n-1} \operatorname{cov} (r, b),$$

where $r$ and $b$ are estimated as 0.30 and 0.34 respectively.

[19] It has to be mentioned that the introduction of $F_t$ (stocks) in relation (6)—and also in other multiple regressions of the shoe manufacturers—gives rise to a significant *positive* sign of the regression coefficient of $F_t$, and to significant coefficients of the other variables as well. The squared correlation coefficient of the relation is 0.759.

[20] See footnote 16.

in his studies on price variables [14] better results than we did in our study on quantity and value variables,[21] although his relations too were subject to asymmetry in the reaction patterns. It may be that the availability of Business Test data on capacity limitations would have improved our results, since in some parts of the period 1956-58 a full-employment situation was reached and might have hampered further increases in production. Besides, some seasonality in the monthly fluctuations for several variables may have been present.

As we pointed out before, other combinations of explanatory variables than those which we considered are conceivable. Still more data are collected each month and the questionnaires are regularly reconsidered for improvements and extensions. Further research is certainly worth-while. An analysis of the relation between results of regressions using B. T. data and of those using corresponding numerical data should also be fruitful.

## 6. CONDITIONAL FORECASTS FOR 1959

The preceding analysis took several months, and in the meantime new data for the twelve months of 1959 became available which we used for prediction purposes. That is, we applied the right-hand sides of the equations (2)-(8) to the data of 1959, to yield $\bar{P}^*_{t+1}$, i.e., the conditional forecast of $\bar{P}_{t+1}$ under the assumption that the values taken by the explanatory variables are as observed. Thus twelve prediction errors, $\bar{P}^*_{t+1} - \bar{P}_{t+1}$, were obtained for each of the seven equations, and they were standardized by dividing them by the least-squares estimated

TABLE 3

FIRST AND SECOND MOMENTS OF STANDARDIZED PREDICTION ERRORS

| | Tanners | | | Shoe Manufacturers | | | |
|---|---|---|---|---|---|---|---|
| | (2) | (3) | (4) | (5) | (6) | (7) | (8) |
| $m_1$ | −0.19 | −0.41 | 0.18 | −0.18 | −0.28 | 0.02 | 0.11 |
| $m_2$ | 0.87 | 0.96 | 1.21 | 2.09 | 2.75 | 1.90 | 1.68 |

standard deviations of the regression disturbances (corrected for loss of degrees of freedom). The means ($m_1$) and the second moments around zero ($m_2$) of the standardized prediction errors are shown in

[21] For references on empirical studies of B. T. data, see Marquardt and Strigel [13]. Also cf. Jochems and Neudecker [7], Jochems and Meijer [8], and Jochems and De Wit [9, 10].

Table 3. If the production planning behavior as pictured by these equations persists during the prediction period, and if the variation is normal, then $m_1$ is normally distributed with zero mean and variance 1/12, and $m_2$ is distributed according to $\chi^2$ with 12 degrees of freedom. We neglect here the random variation of the estimated coefficients. It appears that none of the means are significantly different from zero but that the second moments of the shoe manufacturers' relations tend to be too large. The latter feature points to a larger variation in the prediction than in the sample period, but there are obvious qualifications which need hardly be stated explicitly. Generally speaking, it seems as if the formation of production plans of tanners in 1959 took place in the same way as in 1956-58, but that there were some differences as regards the production planning of shoe manufacturers. The results for 1959 are also illustrated in Figures 1-7.

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# AGGREGATION OVER TIME IN DISTRIBUTED LAG MODELS*

BY YAIR MUNDLAK[1]

## 1. INTRODUCTION

THIS PAPER deals with problems encountered in aggregation over time in models which assume distributed lags. These questions were posed by Ironmonger and Nerlove and in an unpublished paper by the author.[2]

Models leading to distributed lags have been discussed elsewhere and will be introduced here only for future reference.[3] It is assumed that in every period $t$ there is a desired quantity $q_t^*$ of some economic variable, and that is not observed in that period. The observed value is determined by the adjustment equation:

$$(1) \qquad q_t - q_{t-1} = \beta(q_t^* - q_{t-1}) , \qquad 0 \leqq \beta \leqq 1 ,$$

where $\beta$ is the coefficient of adjustment and $q_t$ is the quantity observed in period $t$. The empirical equation derived from (1) is

$$(2) \qquad q_t = \beta q_t^* + (1 - \beta)q_{t-1} .$$

By explaining $q_t^*$ in terms of observable variables, we get an equation which can be estimated from available data. Suppose that

$$q_t^* = \alpha y_t ,$$

where $\alpha$ refers to a long run coefficient and $y$ is an observable variable, then the regression coefficients of

$$(3) \qquad q_t = b_1 y_t + b_2 q_{t-1}$$

can be computed to yield the estimates

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[1] I have benefited from discussions on the subject of this paper at the Econometric Seminar of Yehuda Grunfeld at the Hebrew University, winter, 1960. I am also indebted to Yehuda Grunfeld for reading the manuscript and for making valuable comments.

The presentation of the final draft has been improved as a result of suggestions made by the referee, L. R. Klein, and P Zusman.

[2] D. S. Ironmonger, "A note on the Estimation of Long Run Elasticities, *Journal of Farm Economics*, XLI (1959), 626-32.

Marc Nerlove," On the Estimation of Long Run Elasticities, A Reply," *Journal of Farm Economics*, XLI (1959), 632-40.

Yair Mundlak, "The Demand for Durable Inputs in Moshavim," working paper, Falk Project for Economic Research in Israel, December, 1959 (typed).

[3] Marc Nerlove, "Distributed Lags and Estimation of Long Run Elasticities of Supply and Demand: Theoretical Considerations," *Journal of Farm Economics*, XL (1958) 301-11, and references mentioned there.

$$\hat{\beta} = 1 - b_2$$

$$\hat{\alpha} = \frac{b_1}{1 - b_2} .$$

In this paper we shall trace the consequences of a difference between the period of adjustment for which $\beta$ is defined and the unit period for which the data used in computing the regression are observed. The consequences will be evaluated with respect to the estimation of the adjustment coefficient and the long run coefficient. It will be more convenient to work with (2), where $\alpha$ does not appear. At any point of the development we can substitute for $q_t^*$ and bring in $\alpha$ explicitly. However, this is not necessary. It suffices to note that a solution for $\alpha$ in (3) is equivalent to a reduction of the coefficient of $q_t^*$ in (2) to a value of 1. Any bias in specification or estimation which leads, after reduction, to a different value for that coefficient can be referred to as a bias in obtaining the long run coefficient.

## 2. THE IMPLICATION OF TIME AGGREGATION

It shall be assumed that the adjustment period is a month. There are $k$ months every year. The monthly adjustment equation is

$$q_{t,j} = \beta q_{t,j}^* + (1 - \beta) q_{t,j-1} ; \tag{4}$$

$t$ = year designation, $j$ = month designation,

$j = 1, \cdots, k$, and $(t, 0) = (t - 1, k)$.

The annual variables are defined as

$$Q_t = \sum_j q_{t,j} , \qquad Q_t^* = \sum_j q_{t,j}^* . \tag{5}$$

If for some reason it were desirable to run a regression with annual data, the correct aggregation would be obtained by inserting (4) into (5)

$$Q_t = \beta Q_t^* + (1 - \beta)[Q_{t-1} - (q_{t,k} - q_{t-1,k})] , \tag{6}$$

or

$$Q_t = Q_t^* - \frac{(1 - \beta)}{\beta}(q_{t,k} - q_{t-1,k}) . \tag{7}$$

That means that if the contribution of the second term on the right is relatively small, there is no need to introduce a lagged variable in the empirical equation. The contribution will be small when either $\beta$ or $k$ is large. That means that if either the adjustment is actually rapid or if there are many months in the year, assuming a one month lag in (4), the annual equation should not include a lagged quantity.

In what follows we explore the meaning of a regression having the form of (2) with annual quantities replacing the monthly quantities. In order to do so we should transform (7) to be more similar to (2). Write $q_t$ as a distributed lag

$$(8) \qquad q_{t1} = \frac{\beta}{1-\beta}\sum(1-\beta)^{k+1-j}q_{tj}^* + (1-\beta)^k q_{t-1,k}\,.$$

Substitute (8) into (7)

$$(9) \qquad Q_t = Q_t^* - \sum_j (1-\beta)^{k+1-j}q_{tj}^* + BQ_{t-1,k}\,,$$

where

$$Q_{t-1,k} = kq_{t-1,k}$$

and

$$(10) \qquad B = \frac{1-\beta}{k\beta}[1-(1-\beta)^k]\,.$$

Let $q_{tj}^* = \lambda_{tj}q_t^*$ for $q_t^* = Q_t^*/k$, and hence $\sum_j \lambda_{tj} = k$. Thus we may write

$$(11) \qquad Q_t = (1-C_t)Q_t^* + BQ_{t-1} + Bd_{t-1}\,,$$
$$d_{t-1} = Q_{t-1,k} - Q_{t-1}\,.$$

Equation (11) differs from the form of (2) in that:

1. $C_t$ depends on time and is not a constant in a time series study.
2. There is an additional variable in (11), $d_{t-1}$, which expresses the difference between the annual quantity derived on the basis of the last month of $t-1$ and the actual quantity of last year. The difference gives a measure of the amount of disequilibrium in $t-1$, in the sense that if there were equilibrium, $d_{t-1}$ would disappear.
3. Even if there is an equilibrium in $t-1$, $B$ is not simply the sum of the monthly coefficients. Also $(1-B)$ is not the annual adjustment coefficient. It will be so only in a very special case, which will be investigated later.
4. $B$ tends to disappear for large $k$, since $\lim_{k\to\infty} B = 0$. It will also disappear for $\beta = 1$. Those are the two cases in which (7) yields $Q_t^* = Q_t$.

In the limiting case where $B = 0$ the annual relation does not contain a lagged quantity. Furthermore, the inclusion of a lagged annual quantity is inconsistent with the monthly adjustment model.

The fact that $B = 0$ does not imply that there can be no correlation between $Q_t$ and $Q_{t-1}$. It is well known that economic variables are likely to be serially correlated. Consequently, a relatively high value

for the regression coefficient of the lagged quantity is obtained. This then leads to the conclusion that the adjustment coefficient is low and the long run coefficient is high.

The discussion above indicates that such an inference can be quite erroneous. It is based on the introduction of a variable which is not included in the model but which is highly correlated with the dependent variable. All this is true for the limiting case of $B = 0$. The question, then, is how large is $B$? The value of $B$ depends on both $\beta$ and $k$. In the following table selected values of $\beta$ and $k$ which yield $B \cong 0.1$ are presented.

TABLE 1

VALUES OF $\beta$ AND $k$ WHICH YIELD $B \cong 0.1$

| $\beta$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|
| $k$ | 90 | 40 | 23 | 15 | 10 | 7 | 4 | 3 | 1 |

When $\beta$ is high, values of $k$ leading to a low value of $B$ are relatively small. The implication is clear. Recent empirical findings show low values for $\beta$.[4] We can say that if $\beta$ is in fact high, say 0.9, and the adjustment is made on a quarterly basis, then $B$ will be approximately zero; hence there is no annual adjustment equation. In this case we say that the annual adjustment equation has no meaning, except for the trivial case of $k = 1$, where there is no aggregation. The fact that the estimate of $B$ obtained in regression analysis is relatively high is only an indication of association among the two variables but is not explained in terms of the model. For that matter, one could replace $Q_{t-1}$ by $Q_{t+1}$ and perhaps also obtain high values for estimates of $B$. It would be just as meaningless as the first procedure.

If the adjustment is monthly, then what is said above would actually be valid for values of $\beta > 0.5$. In case of weekly adjustment this would hold true for values of $\beta \geqq 0.2$. This emphasizes that the period of adjustment is the cornerstone of the whole analysis. The model in the form presented above does not tolerate aggregation over many periods. The distortion of results, depends on the true value of $\beta$. The difficulty, however, is in deciding on a priori grounds whether $\beta$ is large or small. Without such a decision, the empirical results do

[4] W. Addison and M. Nerlove, "Statistical Estimation of Long Run Elasticities of Supply and Demand," *Journal of Farm Economics*, XL (1958), 861-81. The reference is made to the results on demand.

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not help much, for we do not know whether we are actually getting an estimate of $B$, or whether $B$ is, in fact, nearly zero and we are getting something else.

The foregoing discussion suggests that some attention should be devoted to the definition of the period of adjustment. If the adjustment is of a discrete nature, perhaps the best procedure would be to study behavior in the market.[5] If there is reason to believe that the adjustment is continuous, the problem is to decide how best to approximate the adjustment process. There exists a complementary relation between $\beta$ and the length of the period. The longer the period, the larger is $\beta$.[6] One thing, however, is clear, if the adjustment is nearly completed in a period that is short relative to a year, annual analysis should not be "dynamic".

In the remainder of the discussion, it is assumed that $B \neq 0$ and that there is no strong effect of serial correlation. The annual equation will be analyzed under various assumptions. The pertinence of this discussion is dictated by what was said above.

*Case* 1.

In the simplest case we assume that $\lambda_{tj} = 1$ for all $t$ and $j$. That is, the desired quantity changes its value only once a year, and such a change occurs in the first month. It then remains unchanged for the rest of the year.

Under that assumption we get

$$C_t = B\,,$$

and consequently

$$Q_t = (1 - B)Q_t^* + BQ_{t-1} + Bd_{t-1}\,. \tag{12}$$

(12) differs from (2) in only one respect, by the inclusion of the additional variable $d_{t-1}$. It will disappear if there is an equilibrium in $t - 1$.[7]

By letting $d_{t-1} = 0$, we get a unique expression for the annual adjustment coefficient, $D$

$$D = \frac{Q_t - Q_{t-1}}{Q_t^* - Q_{t-1}} = (1 - B)\,, \qquad Q_t^* \neq Q_{t-1}\,. \tag{13}$$

[5] Here we follow Ironmonger in making reference to the study by C. H. Berry, G. K. Brinegar and S. Johnson, "Short Run Effects Following Controlled Price Changes: Skim Milk," *Journal of Farm Economics*, XL (1958), 892-903.

[6] This was pointed out by Giora Hanoch.

[7] A value of $d_{t-1} = 0$ can then be considered as a necessary condition for equilibrium in $t - 1$. Generally, it is not a sufficient condition, since there can be compensating variations within the year which will make $q_{t-1,k} = q_{t-1}$. However, under the assumption of Case 1 considered here, this cannot happen since the adjustment takes place in only one direction. It is therefore also a sufficient condition.

For given values $(\beta, k)$, $D$ is uniquely determined.

Admittedly, this is a very special case. When it does occur, and when $k > 1$, there is no need to use distributed lags in time series studies, for it assumes an equilibrium every year. It is still a possible case, but perhaps unlikely, in cross section analysis where it can be assumed that all individuals were in equilibrium in year $t - 1$.

Even in this restricted case, the only way to make inferences about the monthly coefficient is to solve (8) for $\beta$, where $B$ is replaced by its estimate, $\hat{B}$. Obviously, the assumption that $1 - B$ is an estimate of $\beta$ gives an erroneous estimate. The error is

$$(1 - B) - \beta = (1 - \beta)\left[1 - \frac{1}{\beta k} + \frac{(1 - \beta)^k}{\beta k}\right].$$

The error is always positive, it tends to $(1 - \beta)$ as $k$ incleases. Given $k$, it is zero for the extreme values $\beta = 0$ or $\beta = 1$. Whatever error is committed in making inferences about the monthly coefficient, the isolation of $Q_t^*$ by dividing its coefficients by $(1 - B)$ yields the correct estimates of the coefficients in the equation describing $Q_t^*$. That means that the long run elasticities derived by this procedure are correct, even though adjustment coefficient is not.

*Case 2.*

The assumption to be relaxed here is that of equilibrium in the previous year, that is, in $t - 1$. We thus assume $d_{t-1} \neq 0$. In this case it is desirable to estimate (12). However, instead we estimate

$$Q_t = (1 - E)Q_t^* + BQ_{t-1} \tag{14}$$

and omit a variable from the regression. Consequently, the estimate of $B$, $\hat{B}$, will be biased, and a division of the first term by $1 - \hat{B}$ will not give us $Q_t^*$. Before exploring the direction of bias, it may be worthwhile to emphasize the arbitrary nature of the annual adjustment coefficient, defined in (13). By substituting the true expression as given in (12) for $Q_t$ in (13), we get

$$D = (1 - B) + B\frac{Q_{t-1,s} - Q_{t-1}}{Q_t^* - Q_{t-1}}, \qquad Q_t^* \neq Q_{t-1}. \tag{15}$$

It is only when $d_{t-1} = 0$ that $D$ will be independent of the year. Otherwise, the contribution of the second term on the right hand side of (15) depends both on the size of $d_{t-1}$ as well as on the difference $Q_t^* - Q_{t-1}$. That is, if the desired level of the annual qutantity is not much different from $Q_{t-1}$, $D$ can take on large values. It seems therefore that $(1 - B)$ is more meaningful than $D$, since it depends only on $\beta$ and $k$. In fact, for the purpose of studying the adjustment

process, it is most desirable to solve for $\beta$. This, however, requires a knowledge of $k$. It is most likely that such knowledge is not available, and we can solve for $\beta$ only under various assumptions about $k$. But it should be clear that $B$ itself is uniqe for any pair $(\beta, k)$. If these two parameters do not change, that is if there is no structural change, then $(1 - B)$ is a better "macro" parameter. Not only is it uniquely defined in terms of micro parameters, but it can also be defined as

$$(16) \qquad (1 - B) = \frac{Q_t - Q_{t-1,k}}{Q_t^* - Q_{t-1,k}} .$$

It measures the adjustment, not from the value of last year, but from the last month of the last year.

That brings us back to the problem of estimating $B$. At this point we should specify explicitly the statistical model with reference to which the estimation will be discussed. We shall rewrite (2) as

$$(2)' \qquad q_t = \beta q_t^* + (1 - \beta) q_{t-1} + u_t$$

and shall assume that the disturbance $u_t$ is nonautocorrelated and independent of $q_t^*$ and $q_{t-1}$. It is not intended here to explore other possibilities. A discussion of this subject can be found in a recent article by Klein.[8] Under the assumptions of the model least squares estimates of the micro as well as of the macro equations are appropriate and are assumed in the remainder of the discussion.

The bias of $\hat{B}$ is

$$(17) \qquad E(\hat{B} - B) = BP_2 ,$$

where $P_2$ is the least squares regression coefficient of $Q_{t-1}$ in the auxiliary regression.[9]

$$(18) \qquad d_{t-1} = P_0 + P_1 Q_t^* + P_2 Q_{t-1} .$$

$P_2$ can also be interpreted as the relative bias of $\hat{B}$. The direction of the bias depends on the sign of $P_2$. If the gap between $Q_{t-1,k}$ and $Q_{t-1}$ increases with larger values of $Q_{t-1}$, keeping $Q_t^*$ unchanged, $P_2$ will be positive; hence $\hat{B}$ is an overestimate of $B$, and $(1 - \hat{B})$ is an underestimate of $(1 - B)$. It can be shown that this will happen when $m_2$ of the following regression:

$$(19) \qquad Q_{t-1}^* = m_1 Q_t^* + m_2 Q_{t-1}$$

[8] L. R. Klein, "The Estimation of Distributed Lags," *Econometrica*, XXVI (1958), 553-66.

[9] For the meaning of auxiliary regression see Zvi Griliches, "Specification Bias in Estimates of Production Functions," *Journal of Farm Economics*, XXXIX (1957), 8-21, and H. Theil, "Specification Errors and the Estimation of Economic Relationships," *Revue de l'Institut International de Statistique*, XXV (1957), 41-51.

is larger than 1.[10] It is possible to get some notion of (19) from the sample, without having the required monthly data for (18). The estimates of the long run coefficients are subject to another source of bias, that of estimating $(1 - B)$, the coefficient of $Q_t^*$ in (12). That is

$$E(1 - \hat{B}) - (1 - B) = BP_1 ,$$

and this will be positive when $m_1$ in (19) is positive, that is, when there is a trend in the stimulus. The estimates of the long run coefficients which are obtained from the ratio,

$$\frac{1 - \hat{B}}{(1 - \hat{B})} \cong \frac{1 - B - BP_2}{1 - B + BP_1} ,$$

depend on the magnitude and the direction of $P_2$ and $P_1$ or of $m_2$ and $m_1$.

It can be concluded that under the assumption of a once-a-year change in the stimulus which takes place on the first month of the year, we get an annual adjustment equation similar in form to that of the monthly equation and differing from it in one respect, the lagged quantity. The fact that we do not use the correct lagged quantity introduces a specification bias which can be evaluated from the available data.

[10] This was derived as follows: Under the assumption $q_{t,j}^* = q_t^*$ for all $j$ we can write

$$q_{tk} = \beta[1 + (1 - \beta) + \cdots + (1 - \beta)^{k-1}]q_t^* + (1 - \beta)^k q_{t-1,k} .$$

Multiply both sides by $k$ and lag one year

$$Q_{t-1,k} = [1 - (1 - \beta)^k]Q_{t-1}^* + (1 - \beta)^k Q_{t-1,k} .$$

Subtract (12) lagged one year from it to get

$$d_{t-1} = [B - (1 - \beta)^k][Q_{t-1}^* - Q_{t-1,k}] .$$

Utilizing (16), we have

$$d_{t-1} = H(Q_{t-1}^* - Q_{t-1}) ,$$

where

$$H = 1 - \frac{(1 - \beta)^k}{B} .$$

This can also be written as

$$H = 1 - \frac{k(1 - \beta)^{k-1}}{S(k)} ,$$

where

$$S(k) = 1 + (1 - \beta) + \cdots + (1 - \beta)^{k-1} .$$

Since $(1 - \beta)^{k-1}$ is the last term in $S(k)$, $H > 0$. If we use the derived expression for $d_{t-1}$ in the least squares values for $P_1$ and $P_2$, we get the result referred to in the text.

*Case* 3.

When the restriction that $\lambda_{t,j} = 1$ is eliminated we shall get a form which differs from that obtained in Case 2. In order to get a particular form we have to assume something about the distribution of the stimuli within the year. A reasonable departure from Case 2 would be to assume that the stimuli follow some trend within the year, say, in the form

$$q_{tj}^* = q_{t-1,k}^* + g_t^j \,, \tag{20}$$

where $g_t$ is the slope of the trend within year $t$. Substituting (20) in to (9) and noting

$$\sum_{j=1}^{k} j(1-\beta)^{k+1-j} = \frac{1-\beta}{\beta}\left\{k - \frac{1-\beta}{\beta}[1-(1-\beta)^k]\right\} = \frac{(1-\beta)k}{\beta}(1-B) \,,$$

we get

$$Q_t = Q_t^* - BQ_{t-1,k}^* - kg_t\frac{(1-\beta)}{\beta}(1-B) + BQ_{t-1,k} \,, \tag{21}$$

where

$$Q_{t-1,k}^* = kq_{t-1,k}^* \,.$$

But

$$Q_t^* = \sum_j q_{tj}^* = Q_{t-1,k}^* + g_t\frac{k(k+1)}{2} \,;$$

therefore,

$$Q_t = kg_t\left[B\frac{(k+1)}{2} - \frac{(1-\beta)}{\beta}(1-B)\right] + (1-B)Q_t^* + BQ_{t-1,k} \,. \tag{22}$$

Recalling that $Q_{t-1,k} = Q_{t-1} + d_{t-1}$, we notice that (22) differs in form from (12) by having an additional term. If the slope of the within-year trend $g_t$, is the same every year, then this additional term will be absorbed in the constant term of the regression, and what was said in Case 2 would also apply here. This seems to have particular importance for cross section analysis, where it is quite possible that the stimuli change at about the same rate for all individuals.

However, it is possible, and perhaps likely, that the within-year trend of the stimuli would change from year to year. Then we shall have another variable, $g_t$, which has to be included in the regression. In most cases it would probably be difficult to evaluate $g_t$. Its omission from the regression will cause another specification bias. When we are concerned with the estimates of $(1 - B)$, this bias will depend on the relation between $g_t$ and $Q_t^*$, as well as on the sign and magnitude of the coefficient of $g_t$ in (22). That coefficient is likely to be negative.

and increase in absolute value as $k$ increases. It is less clear what the relation between $g_t$ and $Q_t^*$ is likely to be. Perhaps, for high values of $Q_t^*$, the within-year trend is likely to be small, or even negative, and vice-versa for low values of $Q_t^*$. That is to say when the level of stimulus is high, it does not increase much within the year, or even decreases. For if it were otherwise, $Q^*$ would continue to increase, and the particular $Q^*$ that was picked as a high value will not be high. If this were true, then the estimate of $(1 - B)$ will have a positive bias. If the stimuli do not follow a linear trend, the accuracy of the estimates depends on the resultant value $C_t$ in (11) and its relation to the other variables.

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# ECONOMETRIC ESTIMATION WITH TEMPORALLY DEPENDENT DISTURBANCE TERMS*

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## 1. INTRODUCTION

IN THIS PAPER the Wold method [17, 18 (209–213)] of coping with temporally dependent disturbance terms in regression analysis is extended to apply to a set of interdependent regression equations. The applicability of the procedure to econometric estimation of reduced form and structural parameters is also explored.

Wold's method involves the use of least squares to estimate regression coefficients and the application of correction factors to make least squares coefficient variance estimators consistent estimators of asymptotic variances. While the regression coefficient estimators provided by this approach are at least asymptotically unbiased, they are not usually efficient, given the presence of temporally dependent disturbance terms.[2] This loss of efficiency is a defect which possibly can be remedied if one can specify correctly the nature of the process generating the disturbance terms. It may then be possible to obtain asymptotically efficient coefficient estimators and consistent variance estimators.[3] When, however, information regarding the process generating the disturbance terms is lacking or very imperfect and the danger of misspecification is thus real, it may be wise to sacrifice some efficiency in estimation in order to obtain consistent estimators of asymptotic variances. In such situations the results provided by the Wold approach appear useful.

To derive the generalization of Wold's approach, we rewrite a system of regression equations in a form similar to a single equation regression model in Section 2. Alternative specifications of the equation system are also presented, and certain properties of least squares estimators under these specifications are stated. In particular, the variance-covariance matrix of coefficient estimators for a specification which incorporates both structural and temporal dependence is given.

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[1] The initial draft of this paper was written at the Cowles Foundation for Research in Economics, Yale University, under a National Science Foundation Fellowship. In revising the initial draft, the author benefited from discussion with R. L. Basmann, P. J. M. van de Bogaard, A. L. Nagar, H. Theil, and H. Wold. Of course all errors are the sole responsibility of the author.

[2] In [9] it is pointed out that in the case of trigonometric or polynomial regressions, the least squares coefficient estimators are asymptotically efficient.

[3] See [2, 13] for a review of some earlier work. In [7] and [12] some recent results are reported.

In Section 3 Wold's approach is briefly considered in relation to the econometric estimation of reduced form and structural equation parameters. Section 4 contains an application of the procedure to the reduced form equations of a model estimated by Haavelmo [10].

## 2. THE ESTIMATION OF PARAMETERS IN REGRESSION EQUATION SYSTEMS

The system under study is:

$$
\begin{aligned}
y_{1t} &= \pi_{11}z_{1t} + \pi_{12}z_{2t} + \cdots + \pi_{1K}z_{Kt} + u_{1t} \\
y_{2t} &= \pi_{21}z_{1t} + \pi_{22}z_{2t} + \cdots + \pi_{2K}z_{Kt} + u_{2t} \\
&\vdots \quad\quad \vdots \quad\quad\quad\quad \vdots \quad\quad\quad \vdots \qquad\qquad t = 1, 2, \cdots, N, \\
y_{Gt} &= \pi_{G1}z_{1t} + \pi_{G2}z_{2t} + \cdots + \pi_{GK}z_{Kt} + u_{Gt}
\end{aligned} \tag{2.1}
$$

where $y_{1t}, y_{2t}, \cdots, y_{Gt}$ represent observations on $G$ random variables; $z_{1t}, z_{2t}, \cdots, z_{Kt}$ are observations on $K$ nonstochastic variables;[4] and $u_{1t}, u_{2t}, \cdots, u_{Gt}$ are $G$ random disturbance variables. In the present study the system (2.1) is rewritten in such a way that prior results relating to the single equation multivariate regression model are applicable. This is accomplished as follows:[5]

$$
\begin{pmatrix} y_{11} \\ y_{12} \\ \vdots \\ y_{1N} \\ y_{21} \\ \vdots \\ y_{2N} \\ \vdots \\ y_{G1} \\ \vdots \\ y_{GN} \end{pmatrix}
=
\begin{pmatrix}
z & 0_{NK} & \cdots & 0_{NK} \\
0_{NK} & z & & \vdots \\
\vdots & & \ddots & \\
0_{NK} & \cdots & 0_{NK} & z
\end{pmatrix}
\begin{pmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_G \end{pmatrix}
+
\begin{pmatrix} u_{11} \\ u_{12} \\ \vdots \\ u_{1N} \\ u_{21} \\ \vdots \\ u_{2N} \\ \vdots \\ u_{G1} \\ \vdots \\ u_{GN} \end{pmatrix}, \tag{2.2}
$$

or more compactly,

$$
Y = Z\Pi + U, \tag{2.3}
$$

where $Y$ and $U$ are each $GN \times 1$ column vectors, $\Pi$ is a $GK \times 1$ column vector of reduced form coefficients, $0_{NK}$ is a matrix of zeros of size $NK$,

[4] The analysis can be extended to include the case in which some or all of the "independent" variables are stochastic and distributed independently of the disturbance terms.

[5] After hitting upon this way of writing the reduced form system, I discovered that G. E. P. Box in [6] had previously written a system in this form in a consideration of an analysis of variance model with interdependent errors.

and $Z$ is a $GN \times GK$ matrix, partitioned as shown in (2.2), in which $z$ is an $N \times K$ matrix of observations on exogenous variables shown below:

$$z = \begin{bmatrix} z_{11} & \cdots & z_{K1} \\ \vdots & \ddots & \vdots \\ z_{1N} & \cdots & z_{KN} \end{bmatrix}.$$

Four alternative specifications of the model in (2.3) are considered, namely:

(a) $EU = 0$ and

$$\Sigma_a = EUU' = \begin{bmatrix} \sigma I_N & 0_N & \cdots & 0_N \\ 0_N & \sigma I_N & \cdots & 0_N \\ \vdots & \vdots & \ddots & \vdots \\ 0_N & 0_N & \cdots & \sigma I_N \end{bmatrix} = \sigma I_{GN}, \tag{2.4}$$

(b) $EU = 0$ and

$$\Sigma_b = EUU' = \begin{bmatrix} \sigma_{11} I_N & 0_N & \cdots & 0_N \\ 0_N & \sigma_{22} I_N & \cdots & 0_N \\ \vdots & \vdots & \ddots & \vdots \\ 0_N & 0_N & \cdots & \sigma_{GG} I_N \end{bmatrix}, \tag{2.5}$$

(c) $EU = 0$ and

$$\Sigma_c = EUU' = \begin{bmatrix} \sigma_{11} I_N & \sigma_{12} I_N & \cdots & \sigma_{1G} I_N \\ \sigma_{21} I_N & \sigma_{22} I_N & \cdots & \sigma_{2G} I_N \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{G1} I_N & \sigma_{G2} I_N & \cdots & \sigma_{GG} I_N \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1G} \\ \vdots & \ddots & \vdots \\ \sigma_{G1} & \cdots & \sigma_{GG} \end{bmatrix} \otimes I_N, \tag{2.6}$$

where $\otimes$ denotes Kronecker or direct matrix multiplication, and

(d) $EU = 0$ and

$$\Sigma_d = EUU' = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \cdots & \Sigma_{1G} \\ \Sigma_{21} & \Sigma_{22} & \cdots & \Sigma_{2G} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_{G1} & \Sigma_{G2} & \cdots & \Sigma_{GG} \end{bmatrix}, \tag{2.7}$$

where the matrices of size $N \times N$ on the diagonal of $\Sigma_d$ involve the autocovariances of the elements of $U$ and the off-diagonal matrices have elements which are serial covariances. In other words,

$$\Sigma_{ii} = Eu_i u_i' = \begin{bmatrix} \gamma_{(ii)0} & \gamma_{(ii)1} & \cdots & \gamma_{(ii)N-1} \\ \gamma_{(ii)1} & \gamma_{(ii)0} & \cdots & \gamma_{(ii)N-2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{(ii)N-1} & \gamma_{(ii)N-2} & \cdots & \gamma_{(ii)0} \end{bmatrix} = \sigma_{ii} \begin{bmatrix} 1 & \rho_{(ii)1} & \cdots & \rho_{(ii)N-1} \\ \rho_{(ii)1} & 1 & \cdots & \rho_{(ii)N-2} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{(ii)N-1} & \rho_{(ii)N-2} & \cdots & 1 \end{bmatrix}, \tag{2.8}$$

where $u_i$ is an $N \times 1$ column vector, the disturbance term in the $i$th reduced form equation,

$$\gamma_{(ii)k} = Eu_{it}u_{it+k}, \quad \sigma_{ii} = \gamma_{(ii)0}, \quad \rho_{(ii)k} = \gamma_{(ii)k}/\sigma_{ii},$$

and

$$\Sigma_{ij} = Eu_iu_j' = \begin{bmatrix} \gamma_{(ij)0} & \gamma_{(ij)1} & \cdots & \gamma_{(ij)N-1} \\ \gamma_{(ij)1} & \gamma_{(ij)0} & \cdots & \gamma_{(ij)N-2} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{(ij)N-1} & \gamma_{(ij)N-2} & \cdots & \gamma_{(ij)0} \end{bmatrix}$$

(2.9)

$$= \sqrt{\sigma_{ii}\sigma_{jj}} \begin{bmatrix} 1 & \rho_{(ij)1} & \cdots & \rho_{(ij)N-1} \\ \rho_{(ij)1} & 1 & \cdots & \rho_{(ij)N-2} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{(ij)N-1} & \rho_{(ij)N-2} & \cdots & 1 \end{bmatrix},$$

where $u_j$ is an $N \times 1$ column vector, the disturbance term in the $j$th reduced form equation, $\gamma_{(ij)k} = Eu_{it}u_{jt+k}$, $\sigma_{ij} = \gamma_{(ij)0}$, and $\rho_{(ij)k} = \gamma_{(ij)k}/\sqrt{\sigma_{ii}\sigma_{jj}}$. It is assumed that the stochastic process generating the disturbance terms is stationary. Therefore $\gamma_{(ij)k} = \gamma_{(ij)-k}$, for all $i$ and $j$, and $Eu_{it}u_{jt+k} = Eu_{it'}u_{jt'+k}$ for $t \neq t'$ and all $i$ and $j$. The assumption of stationarity also makes the matrices $\Sigma_{ii}$ and $\Sigma_{ij}$ Laurentian.

In (a) the equations of (2.1) are assumed to be structurally and temporally independent and homoscedastic in the sense that the disturbance terms in all equations have the same variance. Specification (b) differs from that in (a) by relaxing the conditions of homoscedasticity.[8] In specification (c) structural, but not temporal, interdependence is introduced. Finally, in specification (d) both structural and temporal interdependence are introduced.

Application of the method of least squares to the system in (2.3) yields the following well-known estimator of $\Pi$:

$$P = (Z'Z)^{-1}Z'Y, \tag{2.10}$$

where $P$ is a $GK \times 1$ column vector estimator of $\Pi$. It is interesting to observe that, with a normality assumption, $P$ is also the maximum-likelihood estimator of $\Pi$ under specifications (a), (b), and (c), but not (d). Further, the same point estimates of the components of $\Pi$ are provided by applying the least squares method to each individual equation of (2.1). This latter observation is apparent if (2.10) is rewritten as

[8] Of course a mere change in units of measurement can reduce specification (b) to that in (a).

$$(2.11)\qquad \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_G \end{bmatrix} = \begin{bmatrix} (z'z)^{-1} & 0_K & \cdots & 0_K \\ 0_K & (z'z)^{-1} & \cdots & 0_K \\ \vdots & \vdots & \ddots & \vdots \\ 0_K & 0_K & \cdots & (z'z)^{-1} \end{bmatrix} \begin{bmatrix} z'y_1 \\ z'y_2 \\ \vdots \\ z'y_G \end{bmatrix},$$

where $p_i$, a $K \times 1$ column vector, is the estimator of the coefficients in the $i$th reduced form equation and $y_i$ is an $N \times 1$ column vector of observations on the $i$th endogenous variable.

The variance-covariance matrix of $P$ is given by

$$(2.12)\qquad E(P - \Pi)(P - \Pi)' = (Z'Z)^{-1}Z'\Sigma Z(Z'Z)^{-1},$$

where $\Sigma = EUU'$ is a $GN \times GN$ matrix.

We shall now set forth some properties of the estimator $P$, given in (3.1), under specifications (a)–(d). First, it is easily shown that under specifications (a)–(d), $EP = \Pi$; that is, $P$ is an unbiased estimator of $\Pi$.[7] Second, as already mentioned, $P$, with a normality assumption, is a maximum-likelihood estimator of $\Pi$ under specifications (a), (b), and (c), but not under (d). Third, the variance-covariance matrices for $P$ under alternative specifications are:

$$(2.13)\qquad V_a = (Z'Z)^{-1}Z'\Sigma_a Z(Z'Z)^{-1} = \begin{bmatrix} (z'z)^{-1} & 0_K & \cdots & 0_K \\ 0_K & (z'z)^{-1} & \cdots & 0_K \\ \vdots & \vdots & \ddots & \vdots \\ 0_K & 0_K & \cdots & (z'z)^{-1} \end{bmatrix} \sigma,$$

$$(2.14)\qquad V_b = (Z'Z)^{-1}Z'\Sigma_b Z(Z'Z)^{-1} = \begin{bmatrix} (z'z)^{-1}\sigma_{11} & 0_K & \cdots & 0_K \\ 0_K & (z'z)^{-1}\sigma_{22} & \cdots & 0_K \\ \vdots & \vdots & \ddots & \vdots \\ 0_K & 0_K & \cdots & (z'z)^{-1}\sigma_{GG} \end{bmatrix},$$

$$(2.15)\qquad V_c = (Z'Z)^{-1}Z'\Sigma_c Z(Z'Z)^{-1} = \begin{bmatrix} (z'z)^{-1}\sigma_{11} & \cdots & (z'z)^{-1}\sigma_{1G} \\ \vdots & \ddots & \vdots \\ (z'z)^{-1}\sigma_{G1} & \cdots & (z'z)^{-1}\sigma_{GG} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1G} \\ \vdots & \ddots & \vdots \\ \sigma_{G1} & \cdots & \sigma_{GG} \end{bmatrix} \otimes (z'z)^{-1},$$

[7] If some of the "independent" variables are stochastic and distributed independently of the disturbance vector, $P$ is a consistent estimator of $\Pi$.

(2.16)
$$V_s = (Z'Z)^{-1}Z'\Sigma_s Z(Z'Z)^{-1} = \begin{bmatrix} (z'z)^{-1}z'\Sigma_{11}z(z'z)^{-1} & \cdots & (z'z)^{-1}z'\Sigma_{1G}z(z'z)^{-1} \\ \vdots & \ddots & \vdots \\ (z'z)^{-1}z'\Sigma_{G1}z(z'z)^{-1} & \cdots & (z'z)^{-1}z'\Sigma_{GG}z(z'z)^{-1} \end{bmatrix}$$

(2.17)
$$= (z'z)^{-1} \boxplus \begin{bmatrix} z'\Sigma_{11}z & \cdots & z'\Sigma_{1G}z \\ \vdots & \ddots & \vdots \\ z'\Sigma_{G1}z & \cdots & z'\Sigma_{GG}z \end{bmatrix} \otimes (z'z)^{-1},$$

where $\boxplus$ denotes "inverse" Kronecker multiplication, the definition of which is obvious on comparing (2.16) and (2.17). This last expression may be brought into a form involving auto and serial correlations if it is noted that:[9]

$$\Sigma_{ij} = \left[\sigma_{ij}I + \sum_{t=1}^{N-1}\gamma_{(ij)t}(C^t + C'^t)\right],$$

and

$$z'\Sigma_{ij}z = \left[\sigma_{ij}z'z + \sum_{t=1}^{N-1}\gamma_{(ij)t}z'(C^t + C'^t)z\right],$$

or

$$z'\Sigma_{ij}z = \left[\sigma_{ij}z'z + 2\begin{pmatrix} \sum_{t=1}^{N-1}\gamma_{(ij)t}c_{(11)t} & \cdots & \sum_{t=1}^{N-1}\gamma_{(ij)t}c_{(1K)t} \\ \vdots & \ddots & \vdots \\ \sum_{t=1}^{N-1}\gamma_{(ij)t}c_{(K1)t} & \cdots & \sum_{t=1}^{N-1}\gamma_{(ij)t}c_{(KK)t} \end{pmatrix}\right]$$

where $C$, an $N \times N$ matrix, is given by

$$C = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & 0 \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & 0 \end{bmatrix},$$

and

$$c_{(pq)t} = \sum_{i=1}^{N-t} z_{pi}z_{qi+t}.$$

Then the $ij$th element (a matrix) of $V_s$ is:

(2.18)
$$E(p_i - \pi_i)(p_j - \pi_j)'$$
$$= \sigma_{ij}(z'z)^{-1} + 2(z'z)^{-1}\begin{bmatrix} \sum_{t=1}^{N-1}\gamma_{(ij)t}c_{(11)t} & \cdots & \sum_{t=1}^{N-1}\gamma_{(ij)t}c_{(1K)t} \\ \vdots & & \vdots \\ \sum_{t=1}^{N-1}\gamma_{(ij)t}c_{(K1)t} & \cdots & \sum_{t=1}^{N-1}\gamma_{(ij)t}c_{(KK)t} \end{bmatrix}(z'z)^{-1}.$$

[9] For a treatment of single equation least squares along similar lines, the reader is referred to [14].

This is, of course, the variance-covariance matrix for $p_i$ and $p_j$, where $p_i$ and $p_j$ are vectors the components of which are estimators of the coefficients in the $i$th and $j$th equations, respectively, of (2.1). In the same way that Wold has suggested correcting least squares standard errors in single equation models, the result in (2.18) can be employed to correct standard errors in multi-equation models.[9] Since these corrections may be large, it is of utmost importance to make them when necessary. As in Wold's case, the correction involves, in a multiplicative fashion, the autocorrelation coefficients of both the disturbance term and of the exogenous variables. Further, it should be noted that the variance-covariance matrix for the coefficient estimators of a single equation of the system involves just the autocorrelation coefficients of the disturbance in that equation.

Since the Wold approach involves the sacrifice of some efficiency, it is of interest to set forth expressions for the generalized variances[10] and to indicate how the joint efficiency of the estimator given in (2.10) may be determined under specifications (a)–(d). The generalized variances are shown below:

| Specification | Generalized Variances |
|---|---|
| (a) | $\lvert (Z'Z)^{-1}Z'\Sigma_a Z(Z'Z)^{-1}\rvert = \lvert (Z'Z)^{-1}\rvert \sigma^{GK} = \lvert z'z\rvert^{-G}\sigma^{GK}$ |
| (b) | $\lvert (Z'Z)^{-1}Z'\Sigma_b Z(Z'Z)^{-1}\rvert = \lvert (Z'Z)^{-1}\rvert \sigma_{11}^{K}\cdots\sigma_{GG}^{K} = \lvert z'z\rvert^{-G}\prod_{i=1}^{G}\sigma_{ii}^{K}$ |
| (c)[a] | $\lvert (Z'Z)^{-1}Z'\Sigma_c Z(Z'Z)^{-1}\rvert = \lvert Z'Z\rvert^{-2}\lvert Z'\Sigma_c Z\rvert$ $= \lvert z'z\rvert^{-G}\begin{vmatrix}\sigma_{11}&\cdots&\sigma_{1G}\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\\ \sigma_{G1}&\cdots&\sigma_{GG}\end{vmatrix}^{K}$ |
| (d)[b] | $\lvert (Z'Z)^{-1}Z'\Sigma_d Z(Z'Z)^{-1}\rvert = \lvert Z'Z\rvert^{-2}\lvert Z'\Sigma_d Z\rvert$ $= \lvert z'z\rvert^{-2G}\begin{vmatrix}z'\Sigma_{11}z&\cdots&z'\Sigma_{1G}z\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot\\ z'\Sigma_{G1}z&\cdots&z'\Sigma_{GG}z\end{vmatrix}$ |

[a] Theorem 10, [3, (348)], was employed in deriving this result.

[b] As shown above, this expression may be brought into a form involving auto and serial correlation coefficients.

[9] It should be noted that in making the correction involved in (2.18) substitution of consistent sample estimates of $\sigma_{ij}$ and $\rho_{(ij)l}$ results in a consistent estimate if it is assumed that $\rho_{(ij)l} = 0$ for $l$ larger than some finite number, say $l_0$, or, more generally, that the disturbances satisfy Wold's assumption $A_3^*$ [18, (210)]. See [5, 11] for a discussion of the difficulties of obtaining a consistent estimator of coefficient estimator variances in the general case.

[10] See [3, (167)] for a geometric interpretation of the generalized variance.

In order to obtain the efficiency of $P$ under specifications (a)–(d), Aitken's approach [1, 13] will be followed, namely, to transform the variables in (2.3) so that the transformed disturbance term has a variance-covariance matrix which is a constant times the unit matrix. That is, write (2.3) as

$$HY = HZ\Pi + HU, \tag{2.19}$$

where $H$ is taken so that

$$EHUU'H = H\Sigma H' = I.$$

The least squares estimator of $\Pi$ in (2.19), $P^*$, is given by

$$\begin{aligned} P^* &= (Z'H'HZ)^{-1}Z'H'HY \\ &= \Pi + (Z'\Sigma^{-1}Z)^{-1}ZH'HU, \end{aligned}$$

and

$$E(P^* - \Pi)(P^* - \Pi)' = (Z'\Sigma^{-1}Z)^{-1}Z'\Sigma^{-1}EUU'\Sigma^{-1}Z(Z'\Sigma^{-1}Z)^{-1},$$
$$E(P^* - \Pi)(P^* - \Pi)' = (Z'\Sigma^{-1}Z)^{-1}.$$

The joint efficiencies may be evaluated by dividing $|(Z'\Sigma^{-1}Z)^{-1}|$ by the generalized variances shown above.

## 3. ESTIMATION OF REDUCED FORM AND STRUCTURAL PARAMETERS

The system (2.1) may be viewed as the *unrestricted* reduced form equations associated with an econometric model containing $G$ endogenous and $K$ exogenous variables.[11] In this instance the results presented above apply. However, neglect of restrictions appearing in the structural equations of the model means that the least squares coefficient estimators will not generally be efficient even if the disturbance terms are temporally independent. On the other hand, if the system (2.1) is interpreted as a *restricted* reduced form system, derived from the structural equations by direct solution,[12] then some restrictions implied by the model are incorporated in the reduced form system and this "information" can be utilized in estimating the reduced form system. The corrections developed above could then be made to obtain consistent estimators of asymptotic variances.

Now one member of the Theil-Basmann class of estimators [4, 16] of the structural coefficients in a particular structural equation,

[11] The presence of both temporally dependent disturbance terms and lagged endogenous variables results in the usual coefficient estimators being inconsistent. See [16] for an extremely interesting consideration of this problem.

[12] That is, the matrix of coefficients of the endogenous variables in the model is inverted and multiplied into that of the coefficients of the exogenous variables *before* the system has been estimated. As a result of this operation the dependence of each reduced form coefficient on various structural coefficients is made explicit.

$$y = Y\gamma + X_0\beta + u\,, \tag{3.1}$$

is defined by,

$$\begin{pmatrix} c \\ b \end{pmatrix} = \begin{bmatrix} \bar{Y}'\bar{Y} & \bar{Y}X_0 \\ X_0'\bar{Y} & X_0'X_0 \end{bmatrix}^{-1} \begin{bmatrix} \bar{Y}' \\ X_0' \end{bmatrix} y\,, \tag{3.2}$$

where $u$ is an $N \times 1$ vector of disturbance terms, $[y\ Y]$ is an $N \times (G^* + 1)$ matrix of observations on the $G^* + 1$ endogenous variables appearing with nonzero coefficients in (3.1), $X_0$ is an $N \times K^*$ matrix of observations on the $K^*$ predetermined variables in (3.1) with nonzero coefficients, and $\bar{Y}$, an $N \times G^*$ matrix of reduced form calculated values of $Y$, is given by

$$\bar{Y} = X(X'X)^{-1}X'Y\,, \tag{3.3}$$

in which $X$ is an $N \times K$ matrix of all predetermined variables in the entire model.

The sampling error, $e$, is given by [15, (226)]:

$$e = \begin{pmatrix} c \\ b \end{pmatrix} - \begin{pmatrix} \gamma \\ \beta \end{pmatrix} = \begin{bmatrix} \dfrac{\bar{Y}'\bar{Y}}{N} & \dfrac{\bar{Y}X_0}{N} \\ \dfrac{X_0'\bar{Y}}{N} & \dfrac{X_0'X_0}{N} \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{\bar{Y}'u}{N} \\ \dfrac{X_0'u}{N} \end{bmatrix}. \tag{3.4}$$

Then, employing Slutsky's Theorem,

$$\operatorname{plim} e = \operatorname{plim} M^{-1} \begin{bmatrix} \operatorname{plim} \dfrac{\bar{Y}'u}{N} \\ \operatorname{plim} \dfrac{X_0'u}{N} \end{bmatrix}, \tag{3.5}$$

in which $M^{-1}$ denotes the inverse matrix appearing in (3.4). It is customary to assume that the first probability limit on the right hand side of (3.5) exists. The other probability limits on the r.h.s. of (3.5) will *not* in general be zero given the presence of temporally dependent disturbance terms. Note that

$$\begin{aligned} \operatorname{plim} \frac{\bar{Y}'u}{N} &= \operatorname{plim} Y'X(X'X)^{-1}\frac{X'u}{N} \\ &= \operatorname{plim} P \operatorname{plim} \frac{X'u}{N}\,, \end{aligned}$$

where $P$ is the least squares estimator of the coefficient matrix of the regression of $Y$ on $X$. Now $\operatorname{plim} e = 0$, and the estimator $(c'\ b')'$ is consistent if $\operatorname{plim} X_0'u/N = \operatorname{plim} X'u/N = 0$. These latter conditions will be fulfilled if the elements of $X$ are nonstochastic and/or generated

by stationary stochastic processes which are independent of that generating the disturbance terms. If the $X$ matrix includes lagged endogenous variables, and if the disturbance terms are temporally dependent, plim $e$ is not in general equal to zero, which in this situation means that the Theil-Basmann estimators are not consistent. It is thus vital in the present context to assume that the elements of $X$ are nonstochastic and/or generated by processes independent of that generating the disturbance terms.

The limiting variance-covariance matrix of the coefficient estimator $(c'\ b')'$ is given by

$$\text{plim } Nee'$$

$$= \text{plim } M^{-1}\begin{bmatrix}\bar{Y}'\\ X_0'\end{bmatrix}\left[\frac{uu'}{N}\right][\bar{Y}\ X_0]M^{-1}$$

$$= \text{plim } M^{-1}\begin{vmatrix}\text{plim } \dfrac{\bar{Y}'uu'\bar{Y}}{N} & \text{plim } \dfrac{\bar{Y}'uu'X_0}{N}\\ \text{plim } \dfrac{X_0'uu'\bar{Y}}{N} & \text{plim } \dfrac{X_0'uu'X_0}{N}\end{vmatrix}\text{plim } M^{-1}$$

$$= \text{plim } M^{-1}\begin{bmatrix}(\text{plim } P')\left(\text{plim}\dfrac{X'uu'X}{N}\right)(\text{plim } P) & (\text{plim } P')\left(\text{plim}\dfrac{X'uu'X_0}{N}\right)\\ \left(\text{plim } \dfrac{X_0'uu'X}{N}\right)(\text{plim } P) & \text{plim } \dfrac{X_0'uu'X_0}{N}\end{bmatrix}$$

$$\times \text{plim } M^{-1}.$$

Wold's analysis [17, 18, (209–212)] establishes that plim $X'uu'X/N$ and plim $X_0'uu'X_0/N$ exist under rather general stochastic assumptions regarding the processes generating $X$, $X_0$, and $u$. Since sample autocorrelation coefficients are consistent estimators, the following estimator,

$$\text{Est. plim } Nee' = s^2M^{-1}\begin{bmatrix}\bar{Y}'\\ X_0'\end{bmatrix}\begin{vmatrix}1 & \hat{\rho}_1 & \cdots & \hat{\rho}_{n-1}\\ \hat{\rho}_1 & 1 & & \vdots\\ \vdots & & \ddots & \hat{\rho}_1\\ \hat{\rho}_{n-1} & \cdots & \hat{\rho}_1 & 1\end{vmatrix}[\bar{Y}\ X_0]M^{-1},$$

where $s^2$ is the usual estimator of the variance of $u$, is a consistent estimator[18] of plim $Nee'$. As in the previous section, this result can be utilized to obtain consistent estimators of limiting variances and covariances of structural coefficient estimators.

## 4. AN ILLUSTRATIVE CALCULATION

To illustrate the importance of taking temporal dependence into

[18] See footnote 8.

account, the results of calculations employing a model constructed and estimated by Haavelmo [10] are presented below. In Haavelmo's model there are two reduced form equations, namely,

$$
\begin{aligned}
c_t &= \pi_{11} z_{1t} + \pi_{12} z_{2t} + u_{1t} , \\
y_t &= \pi_{21} z_{1t} + \pi_{22} z_{2t} + u_{2t} ,
\end{aligned}
\tag{4.1}
$$

where $c_t$ = price-deflated per capita personal consumption expenditures, $y_t$ = price-deflated per capita disposable income, $z_{2t}$ = "gross investment" and $z_{1t} = 1$, a "dummy" variable. The disturbance terms are $u_{1t}$ and $u_{2t}$.

The estimate of the system (4.1) yielded by a straightforward application of the method of least squares[14] is:

$$
\begin{aligned}
c_t &= \underset{(13.635)}{298.555} + \underset{(0.138)}{1.499} z_{2t} + \hat{u}_{1t} , \\
y_t &= \underset{(12.211)}{285.788} + \underset{(0.124)}{2.105} z_{2t} + \hat{u}_{2t} ,
\end{aligned}
\tag{4.2}
$$

where the figures in parentheses are standard errors of coefficient estimators, calculated in the usual manner (the square root of diagonal elements in the estimated variance-covariance matrices of coefficient estimators, $S_{11}(z'z)^{-1}$ and $S_{22}(z'z)^{-1}$).

The residuals, $\hat{u}_{1t}$ and $\hat{u}_{2t}$, $t = 1, 2, \cdots, N$, for the estimated relationships in (4.2) were employed to test for the presence of temporal dependence. The calculated values of the Durbin-Watson [8] test statistic $d$ were found to be 0.661 for the residuals of the equation for $c_t$ and 0.474 for those of the equation for $y_t$. Both values are such as to indicate the presence of temporal dependence of the error terms. Durbin and Watson provide the following bounds for critical values of $d$ for regressions involving one independent variable (in addition to the dummy variable) and 15 observations[15] on each variable: 0.81–1.07 for the 1 per cent level of significance; 0.95–1.23 for the 2.5 per cent level of significance; and 1.08–1.36 for the 5 per cent level of significance. These bounds are appropriate for a "one-sided" test (for positive autocorrelation) at the stated levels of significance. For "two-sided" tests, these bounds may be employed along with the calculated values $d$ and $4 - d$ if the indicated significance levels are doubled.

[14] Since the Haavelmo model is "just-identified", the least squares coefficient estimators would also be maximum-likelihood estimators if temporal dependence were not present.

[15] In Haavelmo's case 13 observations on each variable were employed in estimation. Unfortunately, the tables provided by Durbin and Watson do not extend beyond $N = 15$; the bounds change by a small amount, however, when $N$ changes. Therefore, the bounds for $N = 15$ are satisfactory in this case, where the calculated values of $d$ are small.

Since temporal dependence of the disturbance terms is present, calculations were performed to compute corrected standard errors. For this purpose, the sample autocovariances of the disturbance terms, obtained by using computed residuals, were calculated; they are shown below along with estimated autocorrelation coefficients.

SAMPLE AUTOCOVARIANCES AND AUTOCORRELATIONS FOR RESIDUAL TERMS IN HAAVELMO'S MODEL

| $l$ | $\hat{\gamma}_{(11)l} = \frac{1}{N-l}\sum_{i=1}^{N-l} \hat{u}_{1i}\hat{u}_{1i+l}$ | $\hat{\rho}_{(11)l}$ * | $\hat{\gamma}_{(22)l} = \frac{1}{N-l}\sum_{i=1}^{N-l} \hat{u}_{2i}\hat{u}_{2i+l}$ | $\hat{\rho}_{(22)l}$ * |
|---|---|---|---|---|
| 1 | 132.650 | 0.635 | 122.843 | 0.664 |
| 2 | 98.704 | 0.462 | 80.237 | 0.403 |
| 3 | 33.298 | 0.157 | 27.285 | 0.131 |
| 4 | −59.761 | −0.320 | −17.533 | −0.087 |
| 5 | −98.514 | −0.700 | −41.422 | −0.320 |
| 6 | −142.418 | −0.972 | −99.568 | −0.721 |
| 7 | −149.388 | −0.875 | −140.313 | −0.872 |

* The estimated autocorrelations were computed employing the following expressions:

$$\hat{\rho}_{(ii)l} = \sum_{i=1}^{N-l} \hat{u}_{ii}\hat{u}_{ii+l} \Big/ \sum_{i=1}^{N-l} \hat{u}_{ii}^2 .$$

Because of the smallness of the sample, the calculations were not extended beyond $l = 7$.

The estimates $\hat{\gamma}_{(11)l}$ and $\hat{\gamma}_{(22)l}$, $l = 1, 2, \cdots, 7$, in conjunction with the calculated quantities

$$c_{(ij)l} = \sum_{i=1}^{N-l} z_{ii}z_{ji+l}, \qquad i, j = 1, 2 \quad \text{and} \quad l = 1, 2, \cdots, 7,$$

were employed to estimate diagonal terms in the variance-covariance matrix of the coefficient estimators. The calculations, based on equation (2.18), are shown in summary form below:

$$\text{Est. plim } (p_1 - \pi_1)(p_1 - \pi_1)' = \hat{\sigma}_{11}(z'z)^{-1}\left\{I + 2\begin{bmatrix} \sum_l \tilde{\rho}_{(11)l}c_{(11)l} & \sum_l \tilde{\rho}_{(11)l}c_{(12)l} \\ \sum_l \tilde{\rho}_{(11)l}c_{(21)l} & \sum_l \tilde{\rho}_{(11)l}c_{(22)l} \end{bmatrix}(z'z)^{-1}\right\}. \tag{4.3}$$

As noted in [5, 11], it is not correct to substitute in this last expression autocorrelation coefficients estimated from residuals such as are presented in the table above; it is necessary to fit a correlogram and then to use values of the autocorrelation coefficients obtained from the fitted function [thus the use of $\tilde{\rho}$ rather than $\hat{\rho}$ in (4.3)]. For this purpose, it will be assumed that the disturbances are generated by the following moving average process: $u_t = \varepsilon_t + 0.8\varepsilon_{t-1} + 0.6\varepsilon_{t-2} + 0.4\varepsilon_{t-3}$, where $\varepsilon_t$ is a non-autocorrelated random variable

with zero mean and constant variance. This moving average scheme is characterized by the following autocorrelation coefficients:[16] $\rho_0 = 1$, $\rho_1 = 0.704$, $\rho_2 = 0.426$, $\rho_3 = 0.185$. The fourth and higher order autocorrelation coefficients are equal to zero [18, (161)]. It is seen that these values are close to the lower order autocorrelations computed from both sets of residuals above and thus will be employed in computing the expression in (4.3).[17]

For the consumption equation, we have

$$\hat{\sigma}_{11}(z'z)^{-1} = (253.658)\begin{bmatrix} .7329396 & -.00701333 \\ -.00701333 & .00007498 \end{bmatrix},$$

$$= \begin{bmatrix} 185.916 & -1.779 \\ -1.779 & .019 \end{bmatrix}.$$

The diagonal elements in this last matrix are, of course, the uncorrected estimates of the coefficient estimator variances for the first equation. To get the corrected variances, the following quantities are computed:[18]

$$\sum_{i=1}^{3} \tilde{\rho}_{(11)i}c_{(11)i} = 14.981 \quad \sum_{i=1}^{3} \tilde{\rho}_{(11)i}c_{(12)i} = 1356.241 \quad \sum_{i=1}^{3} \tilde{\rho}_{(11)i}c_{(22)i} = 1287.870\,.$$

On substituting in (4.3) we obtain

$$\begin{bmatrix} 185.916 & -1.779 \\ -1.779 & .019 \end{bmatrix}\left\{I + 2\begin{bmatrix} 14.981 & 1356.241 \\ 1287.870 & 121740.259 \end{bmatrix} \times \begin{bmatrix} .7329396 & -.00701333 \\ -.00701333 & .00007498 \end{bmatrix}\right\}$$

$$= \begin{bmatrix} 411.384 & -6.445 \\ -3.577 & 0.0675 \end{bmatrix}.$$

The elements on the diagonal of this last matrix are the corrected variance estimates of the coefficient estimators in the first equation.

For the income equation,

[16] For convenience of presentation, most of the actual figures employed in the calculations are shown rounded to three places beyond the decimal point.

[17] Since the sample is small, the higher order estimated autocorrelations, computed from few observations, are subject to large sampling error. In other than illustrative calculations, it would be desirable to fit the correlogram using more formal methods.

[18] It is to be noted from the definition of $c_{(pq)i}$ above that $c_{(pq)i} \neq c_{(qp)i}$ and thus $\sum_i c_{(pq)i} \neq \sum_i c_{(qp)i}$. These inequalities, due essentially to end-effects, are responsible for the lack of symmetry in the estimated variance-covariance matrix. In many large samples this effect will not be great; however it does deserve further consideration, particularly in samples which are not large.

$$\hat{\sigma}_{22}(z'z)^{-1} = (203.439)\begin{bmatrix} .7329396 & -.00701333 \\ -.00701333 & .00007498 \end{bmatrix}$$

$$= \begin{bmatrix} 149.108 & -1.427 \\ -1.427 & .015 \end{bmatrix}$$

is the uncorrected estimate of the coefficient estimator variance-covariance matrix. To obtain the corrected matrix, the following quantities are needed:[19]

$$\sum_{i=1}^{3} \tilde{\rho}_{(22)i}c_{(22)i} = 121740.259, \quad \sum_{i=1}^{3} \tilde{\rho}_{(22)i}c_{(12)i} = 1356.241, \quad \sum_{i=1}^{3} \tilde{\rho}_{(22)i}c_{(21)i} = 1287.870.$$

Then substituting in (4.3), as before, we obtain the corrected variance-covariance matrix

$$\begin{bmatrix} 329.938 & -5.169 \\ -8.368 & .054 \end{bmatrix}.$$

The estimated system with corrected standard errors is given by

$$\begin{aligned} c_t &= \underset{(20.283)}{298.555} + \underset{(0.260)}{1.499}z_{2t} + \hat{u}_{1t}\,, \\ y_t &= \underset{(18.164)}{285.788} + \underset{(0.233)}{2.105}z_{2t} + \hat{u}_{2t}\,. \end{aligned} \tag{4.4}$$

On comparing (4.2) and (4.4) it is seen that the corrected standard errors are much larger than their uncorrected counterparts. These results illustrate very nicely the importance of correcting standard errors when temporally dependent disturbances are present, a correction which, of course, affects both tests of hypotheses and construction of confidence and forecast regions[20].

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[20] It is recognized that the small sample properties of this correction procedure is an open question.

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# AN ESTIMATE OF THE TIME HORIZON AND EXPECTED YIELD FOR A SELECTED GROUP OF COMMON SHARES, 1935–1955*

BY ROBERT ORTNER[1]

## 1. THE VALUE OF A COMMON SHARE

IN THEORY the present value of any asset, including common shares, is the sum of the present values of all expected future returns attributable to the asset. This simple concept provides a means of estimating both the time horizon and expected yield exhibited by investors with respect to the shares in which they trade, as well as a foundation of value.

The analysis which follows is based on three assumptions which I should indicate at the outset. First, I shall be working with "going-concern" values. Second, shares are purchased for their value as shares. If we consider the largest industrial corporations listed on the New York Stock Exchange, as I do below, these assumptions are both reasonable and realistic. In essence, they suggest that I am concerned with the value of a small minority interest in these firms. Third, in order to say something meaningful at all, I assume investors behave or attempt to behave rationally. If instead of profit-maximization (or improvement) a wide variety of subjective, non-profit considerations were the goal of investment, systematic analysis would probably be impossible. Thus, I assume that market quotations are a sort of marginal opinion of value.

In the case of a share of common stock, the returns to a (minority) investor may be derived from two sources, dividends and future sale. If $D$ equals each expected future annual dividend, the first of which will arrive one year hence, $M$ equals the expected market price at some future point in time, $n$ the number of years from the present (hereafter referred to as the time horizon), and $i$ the rate at which these sums are to be discounted (or the expected yield), the present value, $V$, of a share is fixed by definition as follows:

$$V = M(1+i)^{-n} + D\frac{1-(1+i)^{-n}}{i} \tag{1}$$

if $D$ is expected to be constant, and

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$$V = M(1+i)^{-n} + D\frac{1-\left(\frac{1+r_d}{1+i}\right)^n}{i-r_d} \tag{2}$$

if the dividend stream beginning with $D$ is expected to grow at the rate $r_d$ per period. Clearly, (2) reduces to (1) when $r_d = 0$. Further, $M$ may be expressed as the product of $V$ and a compound growth factor $(1+r_p)^n$, where $r_p$ indicates the expected rate of growth in price.

If we consider changes in $V$ which accompany changes in the variables on the right side of (1), it may easily be shown, that the relative effects on $V$ of varying $M$ and $D$ depend on $n$. The longer the time horizon the more important is a given proportionate change in $D$ relative to the same proportionate change in $M$. The regression analysis which follows will reflect these relationships. It turns out that the time horizon is longest when the importance of the dividend variable (as indicated by the appropriate regression coefficient) is greatest compared with price considerations.

## 2. DISTRIBUTED LAGS AND EXPECTATIONS

Equations (1) and (2) suggest a statistical function explaining present price in terms of a dividend and an expected price variable. Assume that $Y_0$ is dependent on $X_0$, $X_{-1}$, $X_{-2}$, and so on, and, as Koyck suggests, on a random disturbance. We have, then, according to Koyck's scheme,[3]

$$Y_0 = \alpha\sum_{n=0}^{\infty}\lambda^n X_{-n} + \mu_0 . \tag{3}$$

If (3) is lagged one time period and both sides are multiplied by $\lambda$, then

$$\lambda Y_{-1} = \alpha\sum_{n=1}^{\infty}\lambda^n X_{-n} + \lambda\mu_{-1} . \tag{4}$$

Now subtract (4) from (3), add $\lambda Y_{-1}$ to both sides and the result is

$$Y_0 = \alpha X_0 + \lambda Y_{-1} + \mu_0 - \lambda\mu_{-1} . \tag{5}$$

The alternative form produces, however, a statistical difficulty. Koyck shows that least squares estimates of $\alpha$ and $\lambda$ in (5) are not consistent. The bias results from two sources. The composite error term, $\mu_0 - \lambda\mu_{-1}$ is, by definition, auto-correlated, as $\mu_0 - \lambda\mu_{-1}$ and $\mu_{-1} - \lambda\mu_{-2}$ contain

[3] L. M. Koyck, *Distributed Lags and Investment Analysis* (Amsterdam: North Holland Publishing Co., 1954). Koyck's general approach applies the geometric progression of weights to the variable $X$ beginning $K$ periods before 0. Equation (3) is the simplified version he employs himself, where $K = 0$.

a common term. In addition, $\mu_{-1}$ is not independent of $Y_{-1}$, one of the independent variables.[3]

Another application of distributed lags was made by Cagan and Friedman in their hypothesis on the derivation of expectations. As stated first by Cagan, "The expected rate of change in prices is revised per period of time in proportion to the difference between the actual rate of change in prices and the rate of change that was expected."[4] Friedman used the same mechanism of adjustment in measuring expected income.[5] If we consider, for example, price and expected price,[6] then Cagan's statement defines a coefficient of adjustment, $\alpha$, as

$$(6) \qquad \alpha = \frac{P_0^* - P_{-1}^*}{P_0 - P_{-1}^*},$$

where $P_{-1}^*$ is the expected value of price held at time $-1$ for time 0. This implies

$$(7) \qquad P_0^* = \alpha \sum_{n=0}^{\infty} (1-\alpha)^n P_{-n}.$$

Thus the current expectation may be obtained from a weighted average of past observations in which the weights form a geometric progression begining with $\alpha$ and proceeding with the common ratio $(1 - \alpha)$. This is, of course, a simple application to expectations of Koyck's distributed lag hypothesis.

Still another contribution to the literature of distributed lags has been made recently by Marc Nerlove.[7] Referring to the "traditional distinction between long-and short-run elasticities of demand," Nerlove describes a time-path of adjustment in demand which results from technological or institutional rigidities. Given a set of prices and incomes, and sufficient time for full adjustment to these factors, there will be a unique, long-run equilibrium quantity demanded, which may be designated $X_0^*$. Because of the lag in adjustment, and constant

[3] Koyck forms an auto-regressive scheme, $\mu_0 = \omega\mu_{-1} + \varepsilon_0$, with which he derives consistent estimates for $\alpha$ and $\lambda$ on an assumed value of $\omega$. Klein carried the analysis one step further by suggesting that one might estimate all three parameters, $\alpha$, $\lambda$, and $\omega$, from the sample data with the aid of an iterative process. See "The Estimation of Distributed Lags," *Econometrica*, XXVI (October, 1958), 553.

[4] Philip Cagan, "The Monetary Dynamics of Hyperinflation," in *Studies in the Quantity Theory of Money*, (Chicago: University of Chicago Press, 1956) p. 37.

[5] Milton Friedman, *A Theory of the Consumption Function* (Princeton: Princeton University Press, 1957). See pp. 150-151.

[6] For comparison with Koyck, I have substituted price for Cagan's expected rate of change in price.

[7] Marc Nerlove, *Distributed Lags and Demand Analysis*, U. S. Dept. of Agriculture, Agriculture Handbook No. 141 (June, 1958).

change in the determinants, $X_0^*$ is never observed. Rather, there is a continuous time-path of adjustment of demand toward a changing equilibrium value. The basic differential equation which Nerlove considers (when time is regarded as a continuous variable) is

$$\frac{dX}{dt} = \beta(t)[X_0^* - X_0], \qquad 0 \leq \beta(t) < 1. \tag{8}$$

Nerlove assumes for simplicity that $\beta(t)$ is a constant, $\beta$. If time is taken as a discrete variable, the difference equation equivalent to (8) is

$$X_0 - X_{-1} = \beta(X_0^* - X_{-1}), \qquad 0 \leq \beta < 1. \tag{9}$$

### 3. SUGGESTED FUNCTIONS FOR COMMON SHARES

In order to proceed with some empirical analysis suggested by the preliminary discussion, the first step is to translate equation (2) into a statistical relationship in which the relevant variables are observable. I have already indicated that we may interpret the dividend factor as a stream beginning with the current annual dividend and growing annually at the rate $r_d$. We denote present value, $V$, as the present price, $P_0$. Expected future price, $M$, is of course, not observable. Let $M = P_0(1 + r_p)^n = P_{-1}(P_0/P_{-1})(1 + r_p)^n$ and let $P_0/P_{-1} = 1 + r_{p_0}$. This last expression is observable as the latest change in price. We may use $P_{-1}$ as the second variable, and thus we have

$$P_0 = a_0 + b_0 D_0 + c_0 P_{-1} + \mu_0. \tag{10}$$

If algebraic equivalence with (2) is to be retained, $b_0$ is defined (as before) as $[1-\{(1+r_d)/(1+i)\}^n]/(i-r_d)$ and $c_0$ as $(1+r_{p_0})\{(1+r_p)/(1+i)\}^n$. Customarily, regression coefficients are regarded as constants. However, as the components of $b_0$ and $c_0$ undoubtedly vary from company to company, it will be necessary to interpret $n$ and $i$, when they are computed from the values of the coefficients, as averages. Therefore, the coefficients themselves should similarly be interpreted as averages.

The reader will have noticed that (10) is the same function that would be given by the Koyck distributed lag hypothesis when present price is related to present and past dividends. In the distributed lag function, last year's price, as an independent variable, takes the place of all past dividends. Although they appear the same algebraically, the two are not necessarily equivalent empirically. If $P_{-1}$ sums up the effects of all past dividends, it represents also all other variables which determine price over time. And if the elapsed time between periods 0 and $-1$ is not very great, $P_0$ and $P_{-1}$ are likely to be closely

related even though past dividends are not very important variables. Thus all I suggest with respect to (10) is that $P_{-1}$ should prove significant, and that the relationship between $P_0$ and $P_{-1}$ reflect the expected rate of growth in price, the time horizon, and the expected yield.

To the current dividend and price of the previous period, a constant $a_0$ and an error term are added. In theory, $a_0 = 0$; statistically, it should be non-significant and appear over time (the function will be fitted in cross section) as a random variable with zero mean. On the basis of the values for $b_0$ and $c_0$, $n$ and $i$ may be found. But $b_0$ and $c_0$ also involve $r_d$ and $r_p$. Therefore, some decision must be reached first concerning the derivation of these expectations. In general, if expectations are based on recent and past observations, it seems reasonable to apply greatest weight to current dates and progressively smaller weight to values dated further back in time. If these weights approximate a geometric progression, their common ratio might be found through Koyck's suggestions.

A second equation which might be tried is derived from (10) by relating present price to present dividend and the expected rate of growth in price.[8] This may then be combined with the Cagan-Friedman hypothesis on expectations as given in equation (7). Let $E_0$ be $1 + r_p$. Then we have

$$P_0 = a_0 + b_0 D_0 + c_0' E_0 , \tag{11}$$

and

$$E_0 = \alpha \frac{P_0}{P_{-1}} + (1 - \alpha) E_{-1} . \tag{12}$$

We substitute (12) into (11) and add a random term, to get

$$\begin{aligned} P_0 = \bar{a}_0 + b_0 D_0 - \frac{c_0'}{c_{-1}'}(1 - \alpha) b_{-1} D_{-1} \\ + c_0' \alpha \frac{P_0}{P_{-1}} + \frac{c_0'}{c_{-1}'}(1 - \alpha) P_{-1} + \bar{\mu}_0 . \end{aligned} \tag{13}$$

In retaining equivalence between (11) and (10), $c_0'$ must be defined as $P_{-1}(1 + r_{p_0})\{(1 + r_p)^{n-1}/(1 + i)^n\}$. All regression coefficients, it appears, should be positive except that for $D_{-1}$. $\bar{a}_0$, however, cannot now be regarded as a random variable; it is equal to $a_0 - (c_0'/c_{-1}')(1 - \alpha)a_{-1}$, and as $(c_0'/c_{-1}')(1 - \alpha)$ is the coefficient of $P_{-1}$ and is expected to be positive, $\bar{a}_0$ should be negative.

Now, if we can estimate $\alpha$, we may estimate $1 + r_p$. On Koyck's

[8] That is, use one of the $n$ factors of $1 + r_p$ contained in $c_0$ as an independent variable in addition to the current dividend. The determinant $P_{-1}$ may then be regarded as part of a new coefficient $c_0'$ which is defined after equation (13).

suggestion,

$$1 + r_p = \alpha \sum_{s=0}^{\infty} (1-\alpha)^s \frac{P_{-s}}{P_{-s-1}}, \tag{14}$$

that is, a weighted average of past price relatives. If $\alpha$ falls between .4 and 1.0, four or five past observations should be enough to achieve 90 percent or more of the sum of the weights of the infinite series. The regression coefficients of (13), however, do not permit a determination of $\alpha$. One possibility is to use the results of the same equation for the previous year. In year $-1$, $P_{-1}$ would have as one of its determinants $P_{-1}/P_{-2}$ and the coefficient of this variable would be $c'_{-1}\alpha$. Assuming $\alpha$ to be constant or reasonably stable over time, $c'_0\alpha/c'_{-1}\alpha = c'_0/c'_{-1}$, and the coefficients of (13) may then be separated completely.

A third equation may be derived from Nerlove's rigidity hypothesis. It is possible, perhaps, that present price does not reflect complete adjustment to the present dividend and that (10) should be limited to a definition of present value, $V_0$. Nerlove's ideas suggest a two equation system of the following form:

$$V_0 = b_0 D_0 + c_0 P_{-1}, \tag{15}$$

$$P_0 = \beta V_0 + (1-\beta) P_{-1}. \tag{16}$$

By substituting (15) into (16) and adding a constant and residual, we obtain,

$$P_0 = a_0 + \beta b_0 D_0 + [\beta(c_0 - 1) + 1] P_{-1} + \mu_0, \tag{17}$$

an equation in the same two independent variables as (10).

The rigidity adjustment equation does produce a change, however, when it is applied to the system (11) and (12). Again, if the right side of (11) defines present value, we have:

$$V_0 = b_0 D_0 + c'_0 E_0, \tag{18}$$

$$E_0 = \alpha \frac{P_0}{P_{-1}} + (1-\alpha) E_{-1}, \tag{12}$$

$$P_0 = \beta V_0 + (1-\beta) P_{-1}. \tag{16}$$

The solution for $P_0$ is

$$P_0 = \bar{a}_0 + \beta b_0 D_0 - \frac{c'_0}{c'_{-1}}(1-\alpha)\beta b_{-1} D_{-1} + c'_0 \beta \alpha \frac{P_0}{P_{-1}} + \left[\frac{c'_0}{c'_{-1}}(1-\alpha) + (1-\beta)\right] P_{-1} - \frac{c'_0}{c'_{-1}}(1-\alpha)(1-\beta) P_{-2} + \bar{\mu}_0. \tag{19}$$

Comparing (19) with (13) we find a small change. The only difference in variables is that $P_{-2}$ has been added to those of (13). Again, by

using two equations adjacent in time, we may attempt to separate the various components of the regression coefficients.

Equations (13) and (19), it is hoped, will be helpful in estimating $r_p$. Similar procedures may be followed in introducing the expected rate of growth in dividends. Where $E_0$ now represents $1 + r_d$, we have the following system:

$$P_0 = a_0 + b'_0 E_0 + c_0 P_{-1}\,, \tag{20}$$

$$E_0 = \alpha \frac{D_0}{D_{-1}} + (1 - \alpha) E_{-1}\,, \tag{21}$$

with $b'_0 = D_0[1 - \{(1 + r_d)/(1 + i)\}^n]/\{(i - r_d)(1 + r_d)\}$. We reduce this to

$$P_0 = \bar{a}_0 + b'_0 \alpha \frac{D_0}{D_{-1}} + \left[\frac{b'_0}{b'_{-1}}(1 - \alpha) + c_0\right] P_{-1} - \frac{b'_0}{b'_{-1}}(1 - \alpha) c_{-1} P_{-2} + \bar{\mu}_0\,. \tag{22}$$

Here $\bar{a}_0 = a_0 - (b'_0/b'_{-1})(1 - \alpha) a_{-1}$. By introducing again the notion of rigidity we obtain, as before, the following system:

$$V_0 = b'_0 E_0 + c_0 P_{-1}\,, \tag{23}$$

$$E_0 = \alpha \frac{D_0}{D_{-1}} + (1 - \alpha) E_{-1}\,, \tag{21}$$

$$P_0 = \beta V_0 + (1 - \beta) P_{-1}\,. \tag{16}$$

Reduction produces

$$P_0 = \bar{a}_0 + b'_0 \beta \alpha \frac{D_0}{D_{-1}} + \left[\beta(c_0 - 1) + 1 + \frac{b'_0}{b'_{-1}}(1 - \alpha)\right] P_{-1} - [\beta(c_{-1} - 1) + 1] \frac{b'_0}{b'_{-1}}(1 - \alpha) P_{-2} + \bar{\mu}_0\,. \tag{24}$$

In terms of variables, however, (24) is identical with (22).

In summary, the equations to be tried first are (10), (13), (19), and (22). They will be fitted by least squares in cross section, so that a set of results will be obtained for individual years. In (13), (19), and (22) a simple residual term is added to the regression equation which was derived from a two or three equation model. If an error term were added to each of the structural equations the resulting composite error terms would have been as follows:

$$\bar{\mu}_0 = {}_1\mu_0 - \frac{c'_0}{c'_{-1}}(1 - \alpha)\,{}_1\mu_{-1} + c'_0\,{}_2\mu_0\,, \tag{13}$$

$$\bar{\mu}_0 = \beta\,{}_1\mu_0 + {}_2\mu_0 - \frac{c'_0}{c'_{-1}}(1 - \alpha)({}_1\mu_{-1} + {}_2\mu_{-1}) + \beta c'_0\,{}_2\mu_0\,, \tag{19}$$

$$\bar{\mu}_0 = {}_1\mu_0 + b'_0\,{}_2\mu_0 - \frac{b'_0}{b'_{-1}}(1 - \alpha)\,{}_1\mu_{-1}\,. \tag{22}$$

The number which precedes each $\mu$ indicates the structural equation

to which it belongs (i.e., the first, second, or third equation in the model), and the number following indicates the date. The resulting residuals, while linear, are, in every case, auto-correlated and will serve to restrict the results for the purpose of inference.

## 4. EMPIRICAL RESULTS

The companies included in the analysis represent a selected sample of industrial (manufacturing and trade) corporations listed on the New York Stock Exchange. First, a complete list of industries was taken from the Securities and Exchange Commission study, *Survey of American Listed Corporations, Corporate Profits and Return on Net Worth, 1942–1950*. The industries were ranked both by net worth and sales, and a sample of the industries was drawn beginning with the largest and continuing until at least 75 percent of the entire list in both net worth and sales was included. From each of the selected industries, a sample of companies was drawn by the same procedure. The final group of some 148 companies represents more than two-thirds of net worth and sales of all manufacturing and trade companies whose shares are traded on the Exchange.[9]

The data, market values and dividends, are in aggregate form. Market value is measured as the average of the high and low values for the year, and dividends are measured as total dollars disbursed during the year. Unfortunately, a complete set of information was not available for each company each year. Because lagged price appears in every equation and because it seemed advantageous to maintain comparability among years, twenty companies, for which the price variable was missing in one or more years, were eliminated from the list. The remaining 128 companies are identified in the appendix. In addition, (22) requires the ratio of this year's dividend to last year's; this figure cannot be computed each year for the companies which failed to pay a dividend some time. A second group of thirty-seven companies had to be dropped, therefore, before this equation was fitted. These are also identified in the appendix.

*Case* 1. *Basic.*

Equation (10) was estimated first. The relationship was measured in each of the twenty years between 1936 and 1955; these results are presented in Table 1. Column 1 lists the date of the dependent variable,

[9] The data were collected and kindly made available by Sidney Cottle and Tate Whitman. For a more complete description of the sample, see C. Sidney Cottle and Tate Whitman, *Corporate Earning Power and Market Valuation, 1935–55* (Durham: Duke University Press, 1959), chapters 2 and 3.

columns 2–4 contain the values of the regression coefficients for each year, and column 5 indicates the coefficient of multiple determination. Standard errors of regression coefficients are in parentheses under corresponding estimates. The initial results appear quite encouraging. The $R^2$ values are very high. It was noted above that $a_0$ is expected (in theory) to have a value of zero. When this hypothesis is confronted

TABLE 1

ESTIMATES OF EQUATION (10)

| Date of $P_0$ | $a_0$ | $b_0$ | $c_0$ | $R^2$ |
|---|---|---|---|---|
| 1936 | 2.1302 (3.0818) | 3.1749 (0.5108) | 1.1908 (0.0468) | .9814 |
| 1937 | 15.6339 (5.5091) | −4.9798 (1.0668) | 1.1239 (0.0546) | .9714 |
| 1938 | −4.4665 (2.4398) | 3.5053 (0.5775) | 0.7217 (0.0185) | .9921 |
| 1939 | −0.1999 (3.0002) | 2.9406 (0.8760) | 0.9418 (0.0226) | .9895 |
| 1940 | −2.4953 (3.3481) | 1.3994 (0.4058) | 0.8743 (0.0304) | .9865 |
| 1941 | 3.8893 (2.6840) | −0.2577 (0.4201) | 0.9158 (0.0271) | .9891 |
| 1942 | −3.6765 (1.9527) | 3.0976 (0.4299) | 0.7729 (0.0200) | .9937 |
| 1943 | −3.5800 (3.0102) | 1.1599 (0.7472) | 1.2312 (0.0388) | .9907 |
| 1944 | −2.9217 (2.7260) | 5.1814 (0.6104) | 0.8224 (0.0310) | .9932 |
| 1945 | 9.6928 (3.5712) | 2.5336 (0.7222) | 1.0587 (0.0340) | .9957 |
| 1946 | 10.1680 (4.4385) | 8.7245 (0.9005) | 0.6767 (0.1886) | .9894 |
| 1947 | −7.0770 (3.7829) | 3.0311 (0.7782) | 0.8021 (0.0873) | .9907 |
| 1948 | −0.1252 (4.7484) | −2.2912 (0.5049) | 1.1862 (0.0311) | .9867 |
| 1949 | −0.3212 (3.2077) | 1.8013 (0.2836) | 0.8698 (0.0282) | .9850 |
| 1950 | −1.2186 (3.5819) | 1.7231 (0.1323) | 1.0999 (0.0178) | .9959 |
| 1951 | −0.2789 (11.5974) | 2.1142 (0.7781) | 1.1143 (0.0816) | .9727 |
| 1952 | −7.5904 (3.7634) | 4.4120 (0.4037) | 0.8497 (0.0210) | .9945 |
| 1953 | −5.1677 (7.7382) | −0.6857 (0.7081) | 1.0618 (0.0406) | .9905 |
| 1954 | −16.9162 (9.7321) | 0.7685 (0.6814) | 1.2708 (0.0421) | .9931 |
| 1955 | −41.0203 (9.8362) | 7.1604 (0.8955) | 1.0614 (0.0256) | .9969 |

with these twenty observations, the results appear satisfactory. If we adopt the common criterion of significance that the coefficient be at least twice its own standard error, then the actual values are not, in general, statistically significant. Only four times in twenty did $a_0$ exceed twice its standard error; in about half of the years it was less than its standard error. By the same criterion, $b_0$ and $c_0$ are nearly always significant. In four cases, however, $b_0$ has the "wrong" sign (although two of these are not significantly different from zero). The dividend variable, in general, should have a positive influence on price. Typically, during these twenty years it does. I do not think that the two significantly negative results for $b_0$ should lead to rejection of the theory concerning dividend-price relationships.

Since the total contributions of these variables to present price consist of the products of the independent variables and their respective coefficients, and since prices during this period are generally of the order of twenty times dividends, it appears, overall, that last year's price exerts considerably more influence than the current dividend. As I suggested earlier, this indicates a short horizon. Further, as successive values of $b_0$ and $c_0$ show considerable variation, the results may reflect, in addition, fluctuations in the time horizon. The largest value for $n$ should occur in 1946, where $b_0$ is largest relative to $c_0$, with other relatively high values in 1944, 1952, and 1955. These suppositions are confirmed in Section 6.

*Case 2. Price expectations.*

Equation (13) was computed next. In general, the results are as predicted, but they are disappointing in one important respect.[10] The $R^2$ values are still high. I suggested earlier that for this equation $\bar{a}_0$ should be negative. This result is obtained and is statistically significant in every year. Similarly, the signs of the remaining coefficients are nearly always as expected. However, a satisfactory set of values for $\alpha$, the key to the expected rate of growth in price, could not be found. The reader will recall my comment that the terms of the coefficients might easily be separated if results were taken for two years at a time, provided the values for $\alpha$ were reasonably stable. It turned out, however, that $\alpha$ varied considerably from year to year including a number of negative values, which are inadmissible for these purposes. An attempt was made to find the values by successive approximations, but they continued to fluctuate without converging. It was found then that the last rate of change in price is related to present price

[10] Because of limitations of space the regression results for this and the remaining equations are not included.

and, presumably, to the expected rate of growth in price. But the suggestions of Koyck and Cagan do not lead to a solution for $1 + r_t$ in each year. A further attempt is made below to determine the appropriate pattern of weights by including several observations of past rates of change at each point in time.

*Case 3. Expectations and rigidity.*

Equation (19) was derived from a three equation model which included the expected rate of growth in price and Nerlove's rigidity (lag in adjustment) hypothesis. It seemed likely that if Nerlove's rigidity notion were an important extension, a solution might be found for the expected rate of growth in price. Once again, all the terms of the regression coefficients may be separated for the equation in each year if one of the coefficients of the previous year is used. If we assume now that $\beta$ is stable over time, the coefficient for the current dividend which determines $P_{-1}$ is $\beta b_{-1}$. Then from year zero we have $(c'_0/c'_{-1})(1-\alpha)\beta b_{-1}$. If this term is divided by $\beta b_{-1}$, the result is $(c'_0/c'_{-1})(1-\alpha)$. Finally, in the year zero, $\{(c'_0/c'_{-1})(1-\alpha)(1-\beta)\}/\{(c'_0/c'_{-1})(1-\alpha)\} = (1-\beta)$, so we have $\beta$. The $\beta$ values found in this manner were, in reality, quite stable, but near a value of unity. It would appear that partial adjustment in every period is not, for our purposes, an important consideration. The set of $\alpha$'s which then resulted from this equation, while differing somewhat from the initial estimates made for Case 2, were nevertheless still not helpful; for the most part, the values were either greater than unity or negative.

*Case 4. Dividend expectations.*

Equation (22) was derived earlier by relating present price to the expected rate of growth in dividends and last year's price and by combining this relationship with Cagan's hypothesis on the derivation of expectations. Once again, the results exhibit a high degree of association in terms of $R^2$. On closer examination of the various coefficients, however, it appears that the excellent "fit" is to be attributed almost entirely to last year's price. Equation (22) was fitted 19 times. Of the 19 observations on $b'_0\alpha$ (the coefficient of $D_0/D_{-1}$) only seven are statistically significant. Better results were obtained for the variable $P_{-1}$, but not for $\bar{a}_0$, which was also significant in only seven years out of nineteen. It must be concluded, then, that equation (22) is too unreliable to shed any light on the "market's" expected rate of growth in dividends.

*Case 5. A final trial.*

One more attempt was made to determine the influence of past

rates of change in prices and dividends on present prices, in the hope that the inclusion of more observations might produce a more stable and useful set of results. The data were re-grouped so that for each value of $P_0$, there were observations going back to $P_{-5}$ and $D_{-5}$. By considering changes in expectations over a five-year period and models similar to the ones presented above, I was able to include the last five rates of change in price and the last four rates of change in dividends. In both cases, improvement on the earlier results could not be obtained.

First, with respect to price expectations, it turned out that each of the five price relatives was significant in only about half of the sixteen years in which the equation was fitted. It appears, therefore, that one need not go back beyond five years. The pattern of values from $c'_0\alpha$ to $c'_5\alpha(1-\alpha)^4$, the coefficients for the five price relatives, are highly irregular but generally diminishing. If the results had reflected the Koyck and Cagan schemes, we should have found in each year (at least approximately) that estimated values satisfied

$$\frac{c'_0\alpha(1-\alpha)}{c'_1\alpha} = \frac{c'_0\alpha(1-\alpha)^2}{c'_0\alpha(1-\alpha)} = \frac{c'_0\alpha(1-\alpha)^3}{c'_0\alpha(1-\alpha)^2} = \frac{c'_0\alpha(1-\alpha)^4}{c'_0\alpha(1-\alpha)^3},$$

but this was not the case.

The results for dividends were still less helpful, for it turned out that of the four rates included, $b'_0\alpha$ (the coefficient of $D_0/D_{-1}$) was significant only 5 times in 16, while the others [from $b'_0\alpha(1-\alpha)$ to $b'_4\alpha(1-\alpha)^3$], were significant 5 times for all three. Of these, two had the wrong sign. It is interesting to note, therefore, that the present level of dividends is an important determinant of present price, but the market appears to pay little attention to recent rates of change in dividends.

## 5. CONCLUSIONS ON EXPECTED RATES OF CHANGE

It seems reasonable to assume that expectations for a given economic series should be drawn, at least in part, from recent and past behavior of the series itself. This appears to have been true, during the period covered, for prices but not for dividends.

The results indicate that recent rates of change in price behave, in general, as might be expected. They are significant determinants of present price; the signs of the coefficients are nearly always positive, as predicted; the values of the coefficients, though irregular, do diminish as the observations recede in time.[11] It appears that a weighted average

[11] The means of the coefficients from the most recent to the most distant are as follows: 119.0171, 90.5265, 76.3718, 62.2511, and 86.5204. Curiously, the last turns up again. For the group, however, about half the values were statistically non-significant when all five were included; I would prefer to attribute this unusual value to chance.

of past values, with weights declining, may produce a reasonable set of expected rates of growth in price and, further, it appears that the last five years provide a sufficiently long distribution of lags. As to the pattern of weights, we have found little support for Koyck's geometric progression, year by year, in place of a simpler arithmetic sequence. The weights employed below are, therefore, of the latter form, beginning with 5 applied to the current observation, and ending with 1 for the value five periods back. Because the items to be averaged are a chain of price relatives, I decided, finally, to use the more customary geometric average. The results are contained in Table 2.

In the case of expected dividends, it seems that the findings are entirely negative. A number of explanations appear possible. First, dividend expectations may be unrelated to past observations (which seems hard to believe). Or possibly, the market does form expectations based (at least in part) on recent experience, but then pays little attention to the expected rates of growth in the process of price formation. In the second case, some brief outside considerations are much to the point. John Lintner's study[12] showed that almost all companies (of those analyzed) placed a premium on stability of dividend payments, and gradual growth in the rates of payment. Perhaps the most important determinant of this year's dividend is the last dividend paid; and only that part of an increase in earnings would be considered eligible for payment as dividends that management felt was a permanent increase to the annual flow. There is one condition, then, in which our statistical results are perfectly plausible—a very short time horizon. Here, the expected rate of growth may be zero despite changes over the last five years; and, if such were the case, we should expect to find little influence of past rates of change in dividends on present price. Thus we might have for the market as a whole $r_d = 0$.

A second possible explanation of our results is that the market adheres in its expectation to the long term rate of growth (.0305) in dividends[13] and pays little attention to the variations observed in the last two or three years alone. This would reflect, I think, a long time horizon. The solutions obtained for the time horizon may, therefore, determine which of these values is correct. Finally, it is also plausible for the market to form its expectations regarding dividends

[12] John Lintner, "Distribution of Incomes of Corporations Among Dividends, Retained Earnings, and Taxes," *American Economic Review, Papers and Proceedings of the 68th Annual Meeting*, XLVI (May, 1956), 97.

[13] This figure is computed from the dividend indexes between 1871 and 1937, given in Alfred Cowles, III, *Common Stock Indexes*, Cowles Commission Monograph No. 3 (Bloomington: Principia Press, 1938).

in much the same manner as price and be influenced very little by these expectations. Again, if the time horizon is reasonably short (say from two to five or six years) only a small part of the total return will come from dividends and of this a still smaller part will come from the return due to increases in the dividend stream. Therefore the market may legitimately ignore, to a large extent, the expected rate of growth it holds regarding dividends.

Table 2 summarizes this brief discussion. The first column lists values of $1 + r_p$ computed by the method described above. The second column contains values of $1 + r_d$ by the same method. The last two columns indicate the two suggested constant values for $1 + r_d$.

## 6. THE TIME HORIZON AND EXPECTED YIELD

On the basis of the three estimates of $1 + r_d$, three sets of values for $n$ were computed. From equation (10), we find on substitution of the terms of the coefficient $c_0$ into $b_0$

$$(10') \quad b_0(1+r_p)\left(\frac{1+r_{p_0}}{c_0}\right)^{1/n} + \left(\frac{c_0}{1+r_{p_0}}\right)\left(\frac{1+r_d}{1+r_p}\right)^n - b_0(1+r_d) - 1 = 0 .$$

Given both sets of expectations, $r_p$ and $r_d$, the only unknown quantity is $n$. It was determined by a numerical approximation process. The results, presented in Table 3, are given to three significant digits. The values in column 1 correspond to the expected rates of growth in

TABLE 2

ESTIMATES OF $1 + r_p$ AND $1 + r_d$

| Year | $1 + r_p$ | $1 + r_d$ | | |
|---|---|---|---|---|
| | | (1) | (2) | (3) |
| 1941 | 0.942 | 1.057 | 1.0305 | 1.0000 |
| 1942 | 0.933 | 0.985 | " | " |
| 1943 | 1.037 | 1.004 | " | " |
| 1944 | 1.055 | 1.018 | " | " |
| 1945 | 1.128 | 1.012 | " | " |
| 1946 | 1.146 | 1.064 | " | " |
| 1947 | 1.075 | 1.130 | " | " |
| 1948 | 1.050 | 1.150 | " | " |
| 1949 | 1.024 | 1.170 | " | " |
| 1950 | 1.084 | 1.228 | " | " |
| 1951 | 1.149 | 1.133 | " | " |
| 1952 | 1.147 | 1.084 | " | " |
| 1953 | 1.108 | 1.060 | " | " |
| 1954 | 1.158 | 1.065 | " | " |
| 1955 | 1.179 | 1.100 | " | " |

dividends computed by the weighted geometric mean; $n$ in column 2 is for $1 + r_d = 1.0305$; and in column 3, for $1 + r_d = 1$. In those cases where two solutions were found, both are given. No solution is indicated by *N.S.* The most immediate and striking results are that, overall, it appears to make very little difference which set of values is assigned to $1 + r_d$. And, if these estimates are reasonably accurate, the time horizon, while fluctuating somewhat, has been persistently short.

The results for $n$, further, may be used as a guide in choosing among the three estimates of $1 + r_d$ and, therefore, the time horizons themselves. Again, $r_d = 0$ seems reasonable during the period of a very short time horizon, such as that found for 1941, 1953, 1954, etc. In those cases where $n$ was reasonably large, the long term rate was used —column 2. Thus $r_d = .0305$ seems appropriate in 1944, 1948, and 1952. For intervening periods, column 1 was used. On these considerations a single set of time horizons was selected. The corresponding expected yield was computed from

$$(10'') \qquad 1 + i = \left(\frac{1 + r_{s_0}}{c_0}\right)^{1/n}(1 + r_s)$$

for each $n$. The two sets of figures are presented in Table 4. In 1942, $n = 2.4$ was selected because it was early in the war and a short horizon seems appropriate.

TABLE 3

ESTIMATES OF TIME HORIZON, $n$, FOR DIFFERENT VALUES OF $r_d$

| Year | 1 | 2 | 3 |
|---|---|---|---|
| 1941 | 0.10 | 0.13 | 0.19 |
| 1942 | 3.1 | 1.7, 2.9 | 2.4 |
| 1943 | 1.2 | 1.2 | 1.2 |
| 1944 | 7.0 | 6.8 | 7.4 |
| 1945 | 3.6 | 3.5 | 3.7 |
| 1946 | 26.0 | *N.S.* | *N.S.* |
| 1947 | 2.8, 3.4 | 3.8 | 3.9 |
| 1948 | *N.S.* | 6.9 | 2.6 |
| 1949 | 1.0, 1.9 | 2.2 | 2.3 |
| 1950 | 1.1, 1.9 | 2.1 | 2.2 |
| 1951 | 2.7 | 2.9 | 3.0 |
| 1952 | 6.7 | 8.0 | 9.3 |
| 1953 | 1.1 | 0.68 | 0.48 |
| 1954 | 0.90 | 0.89 | 0.89 |
| 1955 | 13.4 | *N.S.* | *N.S.* |

TABLE 4

TIME HORIZONS AND EXPECTED YIELDS

| Year | $n$ | $i$ |
|---|---|---|
| 1941 | (0.19) | (.0006) |
| 1942 | 2.4 | −.0004 |
| 1943 | 1.2 | .0654 |
| 1944 | 6.8 | .0969 |
| 1945 | 3.6 | .1783 |
| 1946 | 26.0 | .1686 |
| 1947 | 3.4 | .1204 |
| 1948 | (6.9) | (.0306) |
| 1949 | 2.3 | .0865 |
| 1950 | 2.2 | .1518 |
| 1951 | 2.7 | .2038 |
| 1952 | 8.0 | .1822 |
| 1953 | (0.48) | (−.0007) |
| 1954 | 0.89 | .1633 |
| 1955 | 13.4 | .1925 |

* Note: The reader should be advised that the pairs of solutions in parentheses correspond to the negative values of $b_0$ in Table 1. They are approximations whose meaning might be questioned for $b_0 < 0$.

The list of expected yields, confirms the original hypotheses and regression results. The holder of a share of stock may, in general, expect to gain from two sources—dividends and price appreciation. The expected yield, therefore, should be greater than the expected rate of growth in price. What is necessary, then, is that $1 + r_{p_0} > c_0$ in each year. This proved to be the case in all years in which equation (10) was tried (see Table 1) between 1936 and 1955, except in 1937, 1948, and 1953.

## 7. SHARE PRICES OVER TIME

If the foregoing analysis has any validity at all, it should be helpful in offering some explanation of the major turning points in share prices. Based on the above results, we have two variables with which to explain the course of share prices—current dividends and the expected yield. Together with the price series itself, these are included in the accompanying chart. I have also indicated, on the same chart, the values of $n$, and the current yields of a group of high-grade non-callable preferred shares.[14] Their non-callable feature gives them, in a sense, the same time dimension as common shares, and their high

[14] This series was established (in monthly form) and kindly made available to me by Roger F. Murray. The annual values are simply arithmetic means of Murray's monthly series.

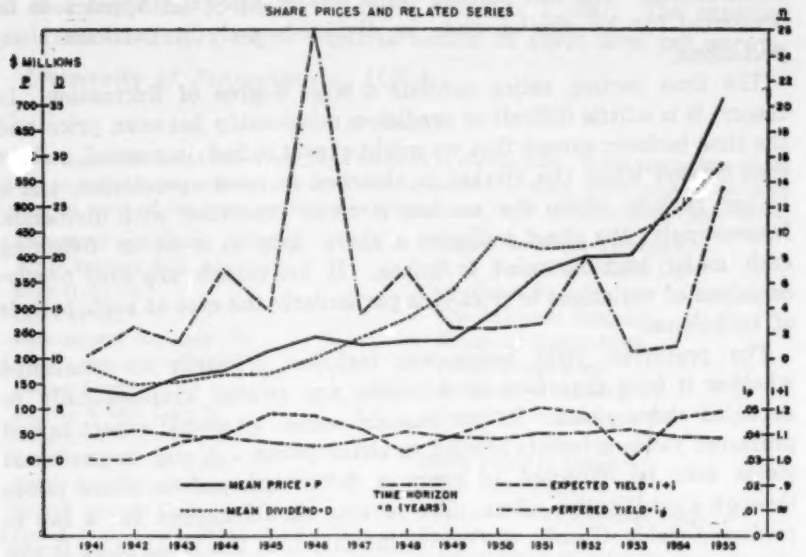


FIGURE 1

quality brings their yields very near those of long-term bonds.

In general, we should expect to find both dividends and expected yields directly related to price over time. The latter relationship, however, may not be quite so simple. A fall in expected yield from, say, 18% to 16% may still leave shares so attractive that they will continue to rise in price. This appears to be the case between 1952 and 1954. Overall, however, the anticipated relationships do hold nicely. Between 1938 and 1955 there are four distinct turning points—1942, 1946, 1949, 1953. In 1942, prices turned upward, together with a slight increase in dividends and a fairly sharp rise in expected yield. Despite the continued rise of dividends in 1946, prices declined slightly. Apparently we may attribute this change to a continued deterioration in expectations. In 1950 prices resumed, once again, their earlier upward movement. There had been an increase in expected yield after 1948, and dividends continued their upward trend. Following these leads, one might have expected a pronounced rise in prices in 1949. Presumably the market had considerable doubt about the future course of dividends (despite the recent excellent increases) and prices. It was another year before the fears diminished sufficiently to permit the beginning of a period of large-scale price increases. In retrospect, as a result, 1949 turns out to be a point of great undervaluation of

these shares. The last turning point was in 1953 and appears to be accounted for by similar rises in dividends and (in particular) expectations.

The time horizon series exhibits a large degree of fluctuation. In theory, it is a little difficult to predict a relationship between price and the time horizon; except that we might expect to find, in general, a short time horizon when the market is absorbed in price speculation, and a longer horizon when the market is more concerned with dividends. Interestingly, the chart indicates a sharp drop in $n$ at or following each major turning point in prices. If individuals are ever overly-conscious of variations in price it is particularly the case at such periods of turbulence.

The preferred yield series was included primarily to determine whether it (and therefore bond yields) are related systematically to expected share yields. In the classical sense, we should expect to find preferred yields inversely related to share prices. A rise in preferred yields may be expected to exert a downward pull on share prices through a substitution effect. The reverse should happen for a fall in preferred yields. Clearly, the curves indicate that this is the case. It was already established that share prices and expected yields move in the same direction; therefore, we have the result that (current) preferred yields and expected share yields should be, and are, inversely related. The two series moved in opposite directions in every year except two, from 1945 to 1946 and from 1950 to 1951. Thus it appears that interest is a worthwhile consideration in the formation of share prices. Indeed, no sense whatever can be made of the course of share prices if something is not said, at the very least, about the expected yield of the shares themselves.

## 8. CONCLUDING COMMENTS

In summary, the analysis presented above is based on the theoretical concept which defines the present value of an asset as the sum of the present values of the expected future returns attributable to the asset. This notion provides interpretations of relevant regression coefficients which, in turn, produce estimates of the time horizon and expected yield. The coefficients, however, involved also the expected rates of growth in dividends and prices. Recent contributions of Koyck, Cagan, Friedman, and Nerlove to the literature of distributed lags and expectations appeared promising, but provided limited assistance in finding the expected rates empirically. The values found for the time horizon were generally small. Only twice in 15 years did $n$ exceed 10; usually

it was less than 5; and three times less than 1 year. The expected yield varied from a slight negative return to about 20% per annum.

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## APPENDIX

The companies listed below are those included in the analysis. In the regressions which involved recent rates of growth in dividends, a number of firms which paid no dividend in one or more of the years covered were eliminated. These are identified by an asterisk.

*Agricultural machinery industry*
- Allis-Chalmers Mfg. Co.
- Case (J. I.) Co.*
- Deere & Co.*
- International Harvester Co.

*Aircraft industry*
- Boeing-Airplane Co.*
- Curtiss-Wright Corp.*
- Douglas Aircraft Co., Inc.*
- Lockheed Aircraft Corp.*
- North American Aviation, Inc.*
- United Aircraft Corp.

*Apparel & accessories chain stores*
- Lerner Stores Corp.
- J. C. Penney Co.

*Automobile industry*
- Chrysler Corp.
- General Motors Corp.

*Automobile parts & accessories industry*
- Bendix Aviation Corp.*
- Borg-Warner Corp.
- Dana Corp.
- Eaton Manufacturing Co.
- The Electric Auto-Lite Co.
- Stewart-Warner Corp.*
- Thompson Products, Inc.

*Building materials industry*
- The Flintkote Company
- Johns-Manville Corp.
- The Ruberoid Co.
- United States Gypsum Co.

*Chemical industry*
- Allied Chemical & Dye Corp.
- American Cyanamid Co.
- The Dow Chemical Co.
- E. I. duPont de Nemours and Co.
- Hercules Powder Co.
- Monsanto Chemical Co.
- Union Carbide Corp.

*Cigarette industry*
- The American Tobacco Co.
- Liggett & Myers Tobacco Co.
- R. J. Reynolds Tobacco Co.

*Copper industry*
- The Anaconda Company
- Kennecott Copper Corp.
- Phelps Dodge Corp.

*Corn products industry*
- Corn Products Refining Co.
- Penick & Ford, Ltd., Inc.

*Dairy products industry*
- The Borden Co.
- National Dairy Products Corp.

*Department stores*
- Allied Stores Corp.*
- Associated Dry Goods Corp.*
- City Stores Co.*
- Federated Department Stores
- Gimbel Brothers, Inc.*
- R. H. Macy and Co., Inc.
- Marshall Field and Co.*
- The May Department Stores Co.

*Distilling industry*
- Distillers Corp.—Segrams Ltd.*
- National Distillers Products Corp.
- Schenley Industries, Inc.*
- Hiram Walker-Gooderham & Worts Ltd.*

*Electrical supplies & equipment industry*
- General Electric Co.
- Westinghouse Electric Corp.

*Ethical drug industry*
- Abbott Laboratories
- Merck and Co., Inc.
- Parke, Davis & Co.

*General industrial machinery and equipment industry*
- American Chain and Cable Co., Inc.
- Blaw-Knox Co.*
- Combustion Engineering, Inc.
- Fairbanks, Morse & Co.
- Ingersoll-Rand Co.
- Link-Belt Co.

Worthington Corp.*

*Glass container industry*
Hazel-Atlas Glass Co.
Owens-Illinois Glass Co.

*Grain mill products industry*
General Mills, Inc.
Pillsbury Mills, Inc.

*Grocery chain stores*
American Stores Company*
First National Stores, Inc.
The Kroger Co.
Safeway Stores, Inc.

*Limited-price variety stores*
W. T. Grant Co.
S. S. Kresge Company
S. H. Kress & Co.
G. C. Murphy Co.
F. W. Woolworth Co.

*Mail-order houses*
Montgomery Ward & Co., Inc.
Sears, Roebuck and Co.

*Meat packing industry*
Armour & Co.*
The Cudahy Packing Co.*
Swift & Co.
Wilson & Co., Inc.*

*Metal container industry*
American Can Co.
Continental Can Co., Inc.

*Non-ferrous metals industry excluding copper*
The American Metal Co., Ltd.*
American Smelting & Refining Co.
The International Nickel Co. of Canada, Ltd.

*Oil industry*
Gulf Oil Corp.
Phillips Petroleum
Shell Oil Co.
Sinclair Oil Corp.
Socony Mobil Oil Co., Inc.
Standard Oil Co. of California
Standard Oil Co. (Indiana)
Standard Oil Co. (New Jersey)
The Texas Co.

*Packaged foods industry*
General Food Corp.
Standard Brands, Inc.

*Paper and allied products industry*
The Champion Paper & Fibre Co.*
Container Corp. of America
Crown Zellerbach Corp.*
Kimberly-Clark Corp.
Marathon Corp.*
The Mead Corp.*
St. Regis Paper Co.*
Scott Paper Co.
Union Bag-Camp Paper Corp.*
West Virginia Pulp and Paper Corp.

*Proprietary drug industry*
American Home Products Corp.
Sterling Drug, Inc.
Vick Chemical Co.

*Radio and television industry*
Sylvania Electric Products, Inc.

*Rubber industry*
The Firestone Tire & Rubber Co.
The B. F. Goodrich Co.*
The Goodyear Tire & Rubber Co.*
United States Rubber Co.*

*Steel industry*
Bethlehem Steel Corp.*
Inland Steel Co.
Jones & Laughlin Steel Corp.*
National Steel Corp.
Republic Steel Corp.*
United States Steel Corp.*

*Textile fabric industry*
Cannon Mills Co.

*Toilet preparations & soap industry*
Colgate-Palmolive Co.
The Procter & Gamble Co.

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# STABILITY CONDITIONS IN INTERNATIONAL TRADE: A GENERAL EQUILIBRIUM ANALYSIS*

By RONALD W. JONES[1]

DEVELOPMENT OF the theory of international trade has relied heavily on general equilibrium forms of analysis, as exemplified in simple models by use of reciprocal demand curves and transformation schedules. The problem of stability in international trade and the closely related problem of the effect of devaluation on the balance of payments have generally been analyzed, however, by partial equilibrium techniques. The resultant stability conditions in models which disregard the income effects of employment changes involve complicated expressions of demand and supply elasticities for imports and exports. In this article I shall analyze the problem by general equilibrium methods to show that the stability conditions can be stated in a simple form.

A general equilibrium expression for stability that isolates the pure demand from the pure supply effects of price changes is developed in the first section of the paper. In Section 2, I shall make explicit the relationship between the stability analysis and two alternative techniques of investigating the effect of changes in exchange rates on the balance of trade. Finally, in Section 3, I compare the general equilibrium results with the familiar Metzler-Robinson-Bickerdike partial equilibrium expression and point out wherein the latter is deficient.

The two country, two commodity, purely competitive model of trade with full employment guaranteed by flexible prices is employed throughout the paper.

## 1. THE GENERAL EQUILIBRIUM STABILITY EXPRESSION

I select as the starting point for my discussion the well-known stability conditions [equation (1)] in the two country trade model expressed in terms of the elasticity of each country's offer curve, $\varepsilon_i$:

$$\varepsilon_1 + \varepsilon_2 > 1 . \tag{1}$$

This is proved by requiring the excess world demand for one of the commodities to be a decreasing function of its relative price. An

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[1] This paper was written under a Ford Foundation grant to the University of Rochester. I am indebted to Professors Klein and Morishima for useful modifications of an earlier draft.

essential ingredient of the proof is Walras' law, which can be reformulated to require the demand for imports in each country to be equal to the value of exports offered in exchange. (This assumption, of course, underlies the construction of an offer curve.) The proof is simple and not reproduced here.

Although conveniently stated, the condition in terms of the $\varepsilon_i$ hides a great deal. As the relative price of a country's imports rises, consumption changes both because of a substitution effect and because of the effect on real income—this latter term compounded by the fact that the country may produce some of the commodity which has risen in price. Domestic production responds to the change in the terms of trade as well. For this reason, I shall decompose the $\varepsilon_i$ into its demand and supply components in order to reformulate the stability conditions. It is sufficient to work with the relationships in country 1 to develop $\varepsilon_1$.

Let country 1 be an importer of $B$ goods (with imports denoted by $M_b$) and an exporter of $A$ goods (with the volume of exports given by $X_a$). Equation (2) defines $M_b$ as the difference between 1's consumption of $B$, $C_b$, and production, $b$:

$$M_b \equiv C_b(p, Y_A) - b\left(a\left(\frac{1}{p}\right)\right). \tag{2}$$

Consumption is a function of $p$, the relative price of $B$, and $Y_A$, income expressed in $A$ units. The production of $B$ in country 1 is related to the production of $A$, $a$, by the transformation schedule and $a$ itself depends on its relative price, $1/p$.

The elasticity of 1's offer curve, $\varepsilon_1$, is obtained by differentiating (2) totally with respect to $p$ and weighting by $-p/M_b$, where by convention the elasticity is defined to be positive. It is, of course, misleading to term $\varepsilon_1$ a "demand" elasticity as it encompasses the reaction to a price change of both demand ($C_b$) and supply ($b$). Therefore I define, in equation (3), the two elasticities for country 1, $\eta_1$ and $e_1$:

$$\eta_1 \equiv -\frac{p}{M_b}\frac{\partial M_b}{\partial p}\bigg|_{\bar{a},\bar{b}},$$

$$e_1 \equiv \frac{1}{pX_a}\frac{\partial X_a}{\partial\left(\frac{1}{p}\right)}\bigg|_{\bar{C}_a,\bar{C}_b}. \tag{3}$$

$\eta_1$, the elasticity of demand for imports, reflects pure demand behavior as production ($a$, $b$) is assumed unchanged. Similarly, $e_1$ shows the

percentage change in exports, relative to a given percentage price change, when the change in consumption of all commodities is ignored. Imports are defined by equation (2) and exports, $X_a$, equal $(a - C_a)$. Thus the definitions in (3) imply the relationships in (4), where $\partial C_b/\partial p|_{\bar{y}}$ is the pure substitution term with real income ($y$) held constant.

$$\eta_1 = -\frac{p}{M_b}\left\{\frac{\partial C_b}{\partial p}\bigg|_{\bar{y}} - C_b\frac{\partial C_b}{\partial Y_A} + \frac{\partial C_b}{\partial Y_A}\frac{\partial Y_A}{\partial p}\bigg|_{\bar{a},\bar{b}}\right\}$$

$$= -\frac{p}{M_b}\left\{\frac{\partial C_b}{\partial p}\bigg|_{\bar{y}} - M_b\frac{\partial C_b}{\partial Y_A}\right\}, \tag{4}$$

$$e_1 = \frac{1}{pX_a}\frac{da}{d\left(\frac{1}{p}\right)}.$$

With these definitions in mind, differentiate $M_b$ in (2) totally and multiply by $-(p/M_b)$ to obtain expression (5) for the elasticity of 1's offer curve, $\varepsilon_1$.

$$\varepsilon_1 = -\frac{p}{M_b}\left\{\frac{\partial C_b}{\partial p}\bigg|_{\bar{y}} - C_b\frac{\partial C_b}{\partial Y_A} + \frac{\partial C_b}{\partial Y_A}\frac{dY_A}{dp}\right\} \tag{5}$$

$$-\frac{p}{M_b}\left\{\frac{db}{da}\frac{da}{d\left(\frac{1}{p}\right)}\frac{1}{p^2}\right\}.$$

$db/da$ is the slope of the transformation schedule and, in equilibrium, is equal to $-1/p$ (unless specialization is complete). $Y_A$ is just $a + bp$, and since $(db/da) = -(1/p)$, $dY_A/dp$ reduces to $b$, the production of importables.[2] Finally, the term $M_b$ in the second expression can be replaced by $(1/p)X_a$ by Walras' law. Introducing all these simplifications into (5) yields equation (6) as the expression for $\varepsilon_1$:

$$\varepsilon_1 = \eta_1 + e_1. \tag{6}$$

The elasticity of a country's offer curve is the sum of its elasticity of demand for imports and elasticity of supply of exports.

A similar relationship holds for country 2 and $\varepsilon_2$. Therefore, equation (1), the stability condition in terms of offer curves, can be rewritten in form (7):

[2] This is another example of what Professor Samuelson terms the Wong-Viner envelope theorem. It is as if, in differentiating $Y_A$, the production of $A$ and $B$ were held constant. The movement along the transformation schedule involves changes of second order magnitude in national income and can thus be neglected. Cf. Paul A. Samuelson, *Foundations of Economic Analysis* (Cambridge: Harvard University Press, 1947), p. 34.

$$(7) \qquad (\eta_1 + \eta_2) + (e_1 + e_2) > 1 .$$

A necessary and sufficient condition for stability is that the sum of the elasticities of demand for imports and the elasticities of supply of exports in the two countries exceed unity.

## 2. STABILITY AND EXCHANGE RATE CHANGES

The preceding analysis is suggestive of a trading world conducted along barter lines. Commodity $A$ is exchanged for commodity $B$ and, if trade is conducted through a monetary medium, this fact is in no way allowed to alter the model. As a reflection of this assumption, $p$, the terms of trade, is the only price affecting demand or supply behavior. Furthermore, each country is assumed to quote prices in the same currency units.

Alternatively, the price of each country's export commodity could have been arbitrarily fixed, say at unity, in terms of its domestic currency. The exchange rate, $r$, defined as the number of units of 1's currency required to purchase one unit of 2's currency, would then need to be explicitly introduced. Designate prices in 2's currency by a prime, $p'_a$, $p'_b$, and let $p_a$, $p_b$ stand for the prices in 1's currency unit. In addition let $p_a$ and $p'_b$ identically (and arbitrarily) be equal to unity. The terms of trade, $p$, are given by $rp'_b/p_a$, which equals $r$. The previous problem of analyzing the effect of changes in $p$ on the excess demand for $B$ can then alternatively be quoted in terms of analyzing the effect of changes in the exchange rate, $r$, on the *potential* balance of trade deficit of country 1. (This is, of course, the excess of the value of 1's demand for imports over 2's demand for imports at the new, devalued exchange rate. The potential deficit is converted into an actual deficit only if trade takes place at the new rate.) The stability problem and the devaluation problem, when cast in these terms, seem to be one and the same. But what can be said if actual trade takes place? Even assuming throughout an *initial* zero trade balance, does Walras' law ensure that a country's *actual* balance of trade remains zero. If not, is the problem of relating exchange rate movements to actual balance of trade deficits and surpluses identical to the stability problem?

The purpose of this section is to explore further these questions by presenting two versions of the "devaluation problem." The first of these requires an analysis of the model when the system is *not* in full general equilibrium. To simplify the exposition I shall assume that each country produces only one commodity ($A$ in 1 and $B$ in 2);

this assumption in no way limits the generality of the argument.

The equations defining general equilibrium in this model consist of two demand equations in each country—for $C_a^{(i)}$ and $C_b^{(i)}$—and a market-clearing equation for one of the commodities. I assume that underlying each demand equation is the assumption (Walras' Law) that each country plans to spend on $A$ and $B$ exactly the value of its current production. This needs to be stressed because, in order to allow trade at disequilibrium prices, I shall assume that each country (or individuals in each country) possesses inventories of the commodity produced in that country. Furthermore, I assume the total value and the composition of these assets (inventories) in no way affect demand behavior—the latter being restricted only by the value of current output. Thus all "real asset effects" are assumed away.

In any state of disequilibrium it is necessary to specify the set of equilibrium equations that are satisfied and those that are not. I shall assume that the demand equations are always fulfilled—demand counts in the market. The difference between production and internal consumption of a commodity is usually referred to as the desired supply of exports; this is the quantity expressed in the offer curve. In disequilibrium I assume this does not equal actual exports. The difference between

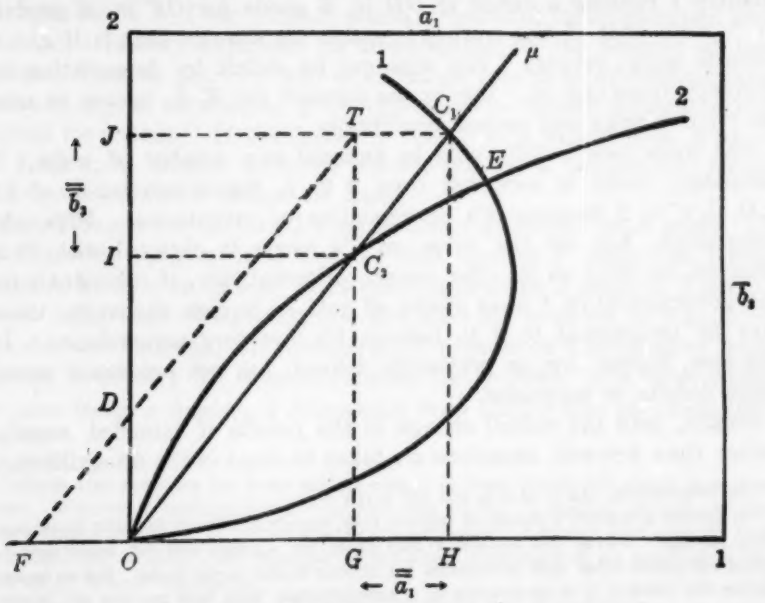


FIGURE 1

desired and actual exports is made up by inventory changes.

The argument is illustrated in Figure 1 of a world production box with total production of $A$ and $B$, $\bar{a}_1$, $\bar{b}_2$, as the sides. Equilibrium is at $E$. The slope of the line $OC_2C_1$ represents a disequilibrium price ratio, (equal to $1/p$ or $1/r$). In accordance with the above assumptions, at the terms of trade $\mu$ each country's demand equations are satisfied. That is, country 1 consumes bundle $C_1$, which is measured from the southeast corner of the box and country 2 consumes bundle $C_2$ relative to its consumption origin (the northwest corner). In order to obtain $OJ$ of $B$ imports, country 1 is willing to export (abstain from consuming) $OH$ in $A$ goods. But actual exports of $A$ goods are assumed to be dictated not by what country 1 would desire to export but by what country 2 demands at $\mu$, amount $OG$. $GH$, or $\bar{\bar{a}}_1$, represents an unintended accumulation of inventories in country 1. Similarly although country 2 is willing to abstain from consuming out of current production only $OI$ of $B$ goods, it runs down its inventories of $B$ by $IJ$ or $\bar{\bar{b}}_2$ in order to satisfy 1's demand.

The point on the figure depicting actual trade flows (relative to the trade origin, $O$) is $T$ and is thus *not* on either country's offer curve (except in equilibrium at $E$). Actual trade takes place, with country 1 running a deficit of $OD$ in $B$ goods (or $OF$ in $A$ goods)[3]. It is clear that if the market is stable (as shown), that is if $\varepsilon_1 + \varepsilon_2$ exceeds unity, country 1 can wipe out its deficit by depreciating its currency (lowering $\mu$). The excess demand for $B$, $\bar{\bar{b}}_2$, serves to raise its relative price and restore equilibrium.

The trade deficit, $OD$, could be financed in a number of ways. If temporary credit is extended from 2 to 1, the accumulation of 1's I. O. U.'s in 2 balances 2's decumulation of inventories. Thus, the composition, but not the value, of 2's assets is changed with, it is assumed, no effect on its offer curve. Alternatively, if individuals (or the government) in 1 have stocks of gold or foreign exchange, these may be transferred to 2 to balance 1's inventory accumulation. In any case, Walras' law, as previously defined, has not prevented actual trade deficits or surpluses[4].

Finally, note the radical change in the results if intended *supply*, rather than demand, equations are taken to count in the disequilibrium

[3] By construction, $OD = IJ = \bar{\bar{b}}_2$ and $OF = GH = \bar{\bar{a}}_1$.

[4] In general a country's excess of imports over exports equals net domestic investment minus savings. Walras' law dictates a zero value for savings, and net investment in capital equipment other than inventories has no role to play in our model. But an import surplus (for country 1) is permissible in a disequilibrium state with positive net investment in the form of unintended inventory accumulation.

situation. At $\mu$, if 1 actually exports $OH$, the amount shown on its offer curve, and 2 exports $OI$, country 1 would have a balance of trade surplus, which might lead to an aggravating appreciation of its currency. I believe this example serves as a warning about the proper interpretation of the offer curves rather than as a practical alternative explanation of the disequilibrium state.

The second treatment of the "devaluation problem" is that offered by James Meade in his *Geometry*[5]. His task is to explore "the relationship(s) between the balance-of-trade deficit and the rate of exchange or barter terms of trade."[6] I shall present a brief algebraic interpretation of Meade's argument to expose the difference between the kind of question he poses and the stability question.

Assume initial equilibrium with zero balance of trade and an equilibrium exchange rate or terms of trade.[7] If each country persists in consuming only the value of its production, any other rate of exchange must be a disequilibrium rate.[8] The relationship between the change in $r$ and the disequilibrium trade deficit has been examined above. Meade's approach is different. Suppose $r$ (or $p$) rises from its equilibrium value. Some extension of credit from one country to the other (or balance of trade deficit or surplus) would allow the new $r$ to be an equilibrium $r$ if the surplus country restricted its consumption and the deficit country expanded its consumption by the amount of the deficit. That is, the new higher exchange rate is an equilibrium rate for some capital movement, *if* demand in each country adjusts to the capital movement.[9] If the higher $r$ results in a capital flow from 1 to 2, (i. e., a balance of trade surplus for 1), a devaluation would improve a country's balance of trade.

Let $T$ represent country 1's trade deficit expressed in $B$ goods. If this becomes positive, it is assumed that country 1 treats the deficit as an increment in income to be spent entirely on $A$ goods or $B$ goods. If we refer back to equation (2), the term $pT$ is added to income in the demand function, $C_b$. Similarly, $T$ appears as a subtraction from income in country 2's demand function for imports. The three equa-

[5] James Meade, *A Geometry of International Trade* (London: Allen and Unwin, 1952) pp. 93–96.

[6] Meade, *op. cit.*, p. 93.

[7] Meade also examines the more general case in which an initial trade deficit or surplus exists. However, to facilitate comparison both with Section I's stability analysis and the next section's discussion of partial equilibrium expressions, I discuss here only the case of initial trade balance. The character of the comparisons made is not sensitive to the zero initial balance assumption.

[8] Unless $\varepsilon_1 + \varepsilon_2 = 1$, in which case many values of $r$ lead to equilibrium.

[9] With reference to footnote 4, savings, rather than investment, is to be non-zero.

tions in (8) summarize the model.

$$
\begin{aligned}
M_a &= M_a\left(\frac{1}{p}, T\right), \\
M_b &= M_b(p, pT), \\
T &= M_b - \frac{1}{p} M_a .
\end{aligned}
\tag{8}
$$

$M_a$, $M_b$, and $T$ are treated as unknowns and $p$ as the parameter. Initially, $T = 0$. Meade wants to find the change in $T$ required to establish a higher $p$ as an equilibrium value.

Differentiate (8), denoting a total derivative with respect to $p$ by a dot over the variable, to obtain (9).

$$
\begin{aligned}
\dot{M}_a \qquad + pm_2\dot{T} &= M_b\varepsilon_2 , \\
\dot{M}_b - \; m_1\dot{T} &= -\frac{M_b}{p}\varepsilon_1 , \\
\frac{1}{p}\dot{M}_a - \dot{M}_b + \qquad \dot{T} &= \frac{M_b}{p} .
\end{aligned}
\tag{9}
$$

Solving for $\dot{T}$ yields equation (10).

$$
\dot{T} = -\frac{M_b}{p}\,\frac{(\varepsilon_1 + \varepsilon_2 - 1)}{(1 - m_1 - m_2)} .
\tag{10}
$$

Devaluation leads to a trade surplus for country 1 if, and only if, $\dot{T}$ is negative, while stability follows if, and only if, $\varepsilon_1 + \varepsilon_2$ exceeds unity.

Thus in Meade's formulation the devaluation problem and the stability problem are essentially different. If the market is stable, a higher exchange rate (or $p$) leads to a trade surplus for the devaluing country if, and only if, the sum of the marginal propensities to import falls short of unity.

This latter condition is, of course, familiar to students of the transfer problem. A transfer from country 1 to country 2 will result in a deterioration of 1's terms of trade (higher $p$) if, and only if, $\sum m_i$ is less than one. And it is precisely this kind of transfer that Meade has in mind.

The difference between the two approaches is now apparent. In my formulation, which leads naturally to the stability conditions, credit can be extended but is not allowed to influence demand. Individuals are faced with a change in the composition of assets and liabilities but not, at any given price, with any change in net worth position. When a country accumulates another country's I. O. U.'s, the increase in assets is offset by a reduction in inventories with no change

in demand. On the other hand, Meade is assuming that credit extension is automatic. The surplus country will voluntarily restrict its purchases and extend credit to the deficit country to establish a new equilibrium at different terms of trade. Meade's criterion may be appropriate in a world of planned deficits and surpluses, but not in a case of *disequilibrium* in the balance of trade where market conditions impinge on prices to bring about adjustment. And it is the latter problem that is equivalent to the question of market stability.

### 3. THE PARTIAL EQUILIBRIUM STABILITY EXPRESSION

The stability expression perhaps most familiar to students of trade theory is the one cited in Metzler's *Survey* article[10] and reproduced in equation (11), in which supply elasticities, $e_i^*$, enter the formulation explicitly and aggregate incomes are held fixed.[11]

$$(11) \qquad \frac{\eta_1^*\eta_2^*(1 + e_1^* + e_2^*) + e_1^*e_2^*(\eta_1^* + \eta_2^* - 1)}{(\eta_1^* + e_2^*)(\eta_2^* + e_1^*)} > 0 .$$

The widespread acceptance of this expression demands that it be analyzed in relation to the two approaches to the exchange rate problem explored in Section 2 and the stability treatment of Section 1.

Clearly (11) and (7) do not seem to be saying the same thing, especially in regard to the role of supply elasticities. If $\sum \eta_i^*$ falls short of unity in (11), stability can still be ensured if supply elasticities, $e_i^*$, are sufficiently *small*. This ambiguous role of supply elasticities is not present in the general equilibrium expression (7), where it is obvious that *large*, not small, values of supply elasticities contribute to stability regardless of the value of the demand elasticities, $\eta_i$.

Two basic steps are involved in deriving (11). In the first, the

[10] Lloyd A. Metzler, "The Theory of International Trade," *A Survey of Contemporary Economics*, Vol. I (Philadelphia: The Blakiston Company, 1948), p. 226. The literature dates back to C. F. Bickerdike, "The Instability of Foreign Exchange," *Economic Journal*, XXX (1920), and expression (11) is derived by, among others, Joan Robinson, "The Foreign Exchanges," in *Essays in the Theory of Employment*, Gottfried Haberler, "The Market for Foreign Exchange and the Stability of the Balance of Payments: A Theoretical Analysis," *Kyklos*, III (1949), and, most recently, by S. S. Alexander, "Effects of a Devaluation: A Simplified Synthesis of Elasticities and Absorption Approaches," *American Economic Review*, XLIX (1959).

[11] The elasticities appearing in (11) are starred to distinguish them from the $\eta_i$ and $e_i$ developed earlier in the paper. $\eta_i^*$ ($e_i^*$) is the percentage change in demand for imports (supply of exports) relative to a given percentage change in price evaluated in the importing country's (exporting country's) currency. The expression assumes a zero initial balance to facilitate comparison with the earlier expressions (7) and (10).

impact of a change in the exchange rate upon the price of exports in each country (evaluated in the currency of the exporting country) is determined by differentiating the market-clearing equations for each commodity. For example, $M_b$ as a function of the price of $B$ in country 1 ($rp_b'$) is set equal to $X_b$ as a function of the supply price in country 2 ($p_b'$). The second step involves differentiating country 1's balance of trade equation and substituting for these price changes. If trade is initially balanced, the devaluation leads to a surplus in country 1 if, and only if, (11) is satisfied.

How does this procedure relate to the two versions of the "devaluation problem" of Section 2? Differentiating the two market-clearing equations with respect to the exchange rate implies that equilibrium is reached both before and after the devaluation—a market equilibrium exists for any exchange rate. This is reminiscent of Meade's treatment discussed in Section 2. Meade, however, explicitly allows the emerging surplus or deficit, acting as an income transfer, to influence demand behavior through the marginal propensities to import, whereas incomes are assumed unaltered in developing (11). In this respect, the partial-equilibrium approach suggests the *dis*equilibrium analysis of the first part of Section 2. It turns out that the assumptions leading to the partial equilibrium expression (11) are consistent with *neither* of the approaches previously discussed but involve a curious blend of parts of both.

This difficulty with the partial-equilibrium analysis is disguised by the ease with which the two market-clearing equations are differentiated to yield the change in the prices of *each* commodity (relative to the exchange rate). One of the virtues of the general equilibrium approach is its explicit recognition that unless separate monetary assumptions are made the system can be solved for only *one* relative price in a two-commodity model. By adopting Section 2's procedure of letting the price of exports in the exporting country's currency unit serve as *numéraire*, the $e_i^*$ become infinite by definition, and the stability condition reduces to $\sum \eta_i^* > 1$. Alternatively, if demand and supply functions are defined in terms of relative rather than absolute prices, the $e_i^*$ are no longer infinite by definition; the procedure leading to (11) also results in $\sum \eta_i^*$ being greater than unity as the stability condition.[12] That is, the $\eta_i^*$ turn out to be the elasticities of the offer

[12] This result follows if the trade balance is expressed as the difference in the value of imports *demanded* in each country. Once these demands are expressed in terms of relative prices, it is not necessary to use the market-clearing equations to solve for absolute prices. Since Walras' law in this case states that $e_1^* = \eta_1^* - 1$, the markets cannot be cleared before and after the devaluation unless $\sum \eta_i^* = 1$.

curves, $\varepsilon_i$, and we return to the beginning equation (1).

To restore the usefulness of the Metzler expression it is necessary to introduce a third, nontraded commodity serving as *numéraire* in each country and explicitly assume that all cross-effects of a change in the price of $A$ on the demand and supply for $B$ and vice versa are zero, and income is held fixed. Such a procedure clearly indicates the *partial*-equilibrium nature of the analysis leading to (11).

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# THE CAPITAL STRUCTURE OF FIRMS AND THE RISK OF FAILURE*

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THIS PAPER is an investigation of the effects of default risk on the portfolio policies of lenders and the capital structure of firms under conditions of perfect competition and constant returns to scale, interpreted in a stochastic sense. We shall develop a highly simplified model of the process by which firms obtain financing, employing the assumptions of the complete information-measurable utility theory; within the context of this model, we shall establish certain propositions concerning the impact of default risk on choices by lenders and entrepreneurs.

The problems studied in this paper arose from an attempt to reformulate Kalecki's "principle of increasing risk" in a consistent, rigorous fashion, when default risk is introduced explicitly into a model in which decision makers follow the rule of maximizing expected utility. In Part 1, which deals with entrepreneurial behavior under uncertainty, we show that Kalecki's principle, stated in terms of the concepts of this paper, holds even when loan markets are perfect and the probability of default positive. It is further demonstrated that under these assumptions the existence of a determinate finite optimal size of firm implies that the entrepreneur is a risk averter, and that this condition is not vacuous.

Part 2 is concerned with the effects of default risk on the portfolio policies of lenders. It is shown that diversification of note holdings by an expected-income-maximizing lender operating in the new loans market may be rationalized. This result may be contrasted with Markowitz' conclusion for the "second hand" securities market that expected-income-maximizing investors will hold portfolios consisting entirely of that security offering the highest expected rate of return (or any probability distribution over securities offering the same (maxi-

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[1] The topic of this paper was suggested to the writer by Professor Leonid Hurwicz, who read several earlier drafts and made a number of criticisms and suggestions that are gratefully acknowledged. Professor Stanley Reiter has also been extremely helpful in clarifying the problems here investigated. I would also like to express my appreciation to the referee, who suggested several alterations that add considerably to the logical structure of the argument. This paper was financed in part by the Office of Naval Research, under Contract Nonr 1100 (16) (NR 047-026).

mum) expected rate of return).[2] In addition, we shall prove that the existence of "capital rationing" does not necessarily imply risk aversion on the part of lenders. Finally, we shall attempt to introduce informational difficulties into the decision problems faced by lenders and to analyze the consequences for portfolio choices.

In this reformulation of Kalecki's problem it is assumed: (1) that the beliefs of decision makers (entrepreneurs and lenders) concerning the occurrence of rates of return for firms may be characterized by continuous subjective probability distributions with finite first and second moments, and (2) that decision makers are concerned solely with income payoffs from their decisions. Income is treated as a random variable, and the preference ordering of a decision maker is defined with respect to probability distributions over income. For any given set of probability distributions over income for a decision maker, a distinction is drawn between "admissible" and "inadmissible" distributions. This distinction is based upon the concept of stochastic dominance, which may be defined for the case of continuous distributions as follows:

*Definition.* A probability density function, $g(z)$, is said to be stochastically dominated by a probability density function, $g^*(z)$, if, and only if, the corresponding distribution functions, $G(z)$ and $G^*(z)$, satisfy:[3]

$$G^*(z) = \int_{-\infty}^{z} g^*(t)dt \leqq \int_{-\infty}^{z} g(t)dt = G(z) ,$$

for all $z$, and the strict inequality applies for at least one value of $z$.

It is clear that stochastic dominance is a transitive relation, so that for a given set of probability distributions over income an "inadmissible" member of the set may now be defined as a distribution that is stochastically dominated by at least one member of the set, while an "admissible" member is defined as a distribution not stochastically dominated by any member of the set. The terms "admissible" and "inadmissible" are used because it is natural to assume that in any choice problem decision makers will consider only admissible distributions in determining an "optimal" choice, since decision makers prefer more to less of whatever they prefer, if we assume nonsaturation of wants.[4] Within the set of admissible distributions, it is

[2] See Harry Markowitz, *Portfolio Selection: Efficient Diversification of Investments*, Cowles Foundation Monograph No. 16 (New York: John Wiley and Sons, Inc., 1959).

[3] The phrases "$g^*(z)$ dominates $g(z)$" and "$G^*(z)$ dominates $G(z)$" are used interchangeably throughout this paper.

[4] See L. J. Savage, *The Foundations of Statistics* (New York: John Wiley and Sons, Inc., 1954), p. 165, for a similar distinction in the minimax case.

assumed that the preference ordering of the decision maker may be represented by a measurable utility function, and that expected utility is a differentiable function of the expected value and variance of income only. That is, higher moments of the probability distribution over income are ignored.[5]

Financing of a firm is assumed to consist of equity capital supplied by an entrepreneur, and loan capital obtained through the sale to lenders of one dollar one-period unsecured notes.[6] All financial transactions are assumed to occur at the beginning of a period, and default occurs at the end of a period if the firm's assets are less than the face value of notes outstanding.[7]

## 1. ENTREPRENEURIAL BEHAVIOR UNDER UNCERTAINTY

We first investigate, under simplified conditions, the problem of borrower (entrepreneurial) behavior under uncertainty.

Let

$m_i$ = the equity of the $i$th firm, $i = 1, 2, \cdots, n$,

$X_i$ = the number of one dollar one-period unsecured notes issued by the $i$th firm,

$p_i$ = the price per one dollar note issued by the $i$th firm,

$A_i \equiv m_i + p_i X_i$ = the assets held by the $i$th firm at the beginning of the period,

$B_i$ = the assets held by the $i$th firm at the end of the period,

$r_i \equiv (B_i - A_i)/A_i$ = the rate of return on assets earned by the $i$th firm during the period,

$d_i \equiv X_i/A_i - 1$ = the "default rate of return" for the $i$th firm, that is, for $r_i = d_i$, $B_i = X_i$, and for $r_i < d_i$, default occurs,

$y_i \equiv \begin{cases} A_i r_i - (1 - p_i)X_i & \text{for } r_i \geqq d_i \\ -m_i & \text{for } r_i \leqq d_i \end{cases}$ = the income of the $i$th firm for the period.

Kalecki's principle of increasing risk states that if the rate of return earned on assets by a firm operating under constant returns and perfect competition is considered a variable subject to uncertainty, then

[5] This assumption may be viewed in terms of a Taylor series approximation in which the utility function is expanded about the expected value of income, the expectation of the series is taken, and terms of higher order than the second are dropped. In effect, the analysis of this paper is restricted to linear or quadratic utility functions. See J. Marschak, "Optimal Investment of a Firm" (abstract), *Econometrica*, XIX (April, 1950), p. 176.

[6] It is further assumed that all of the equity capital for a particular firm is supplied by one particular entrepreneur, so that problems relating to the existence of a group utility function and to financial transactions among entrepreneurs are ignored.

[7] This assumption is equivalent to the assumption that all firms are incorporated.

the risk associated with operation of the firm, as measured by a cardinal risk rate, is an increasing function of size of firm. From this principle, Kalecki argued the possibility of determinacy of size of firm under constant returns and perfect competition. Limitations on size of firm arise from aversion to risk on the part of the entrepreneur and/or from capital limitations (or increasing cost of capital) imposed by lenders or "outside" suppliers of equity capital.[8] In what follows that principle is restated in terms of the concepts and notation of this paper.

We first consider a case in which it is assumed that the entrepreneur holds all of his assets in the form of equity in a particular firm. In this case, income for the entrepreneur equals income for the firm, so that utility for the entrepreneur, $u$, may be written as:[9]

$$u = u(y) ,$$

and, by assumption,

$$Eu = \psi(Ey, \sigma_y^2) .$$

Let $f(r; A)$ denote the subjective probability distribution held by the entrepreneur over rates of return earned by the firm. In general, the size of firm, $A$, is a parameter of this distribution. We now define the following terms:

(1) By "stochastic constant returns" is meant a situation in which the subjective probability distribution over rates of return earned by the firm is independent of size of firm, that is,

$$f(r; A') = f(r; A'') = f(r) \qquad \text{for all } A', A'' > 0 .$$

(2) By a "perfect loan market" is meant a market for notes in which the price of notes issued by a firm is a fixed constant, independent of the number of notes issued, that is,

$$\frac{dp}{dX} = 0 \qquad \text{for all } X \geqq 0 .$$

The principle of increasing risk for entrepreneurs may now be stated

[8] See M. Kalecki, "The Principle of Increasing Risk," *Economica*, IV (November, 1937), pp. 440-447. For commentaries on the Kalecki principle, see J. Steindl, "On Risk," *Oxford Economic Papers*, No. 5 (June, 1941) and "Capitalist Enterprise and Risk," *Oxford Economic Papers*, No. 7 (March, 1945); and Kenneth Boulding, *A Reconstruction of Economics* (New York: John Wiley and Sons, 1950), pp. 126-127.

[9] Since in the following section all variables refer to the same firm, subscripts have been omitted.

in terms of the above notation:[10]

LEMMA 1. *The Principle of Increasing Risk—Loan Capital: If a firm operates with a fixed amount of equity capital under the condition of stochastic constant returns and faces a perfect loan market,* $0 < p \leqq 1$, *and if it is further assumed that* $Er$ *and* $\sigma_r^2$ *exist, where* $\sigma_r^2 > 0$, $Er > (1-p)/p$ *(variance of rate of return positive, and expected rate of return exceeding interest rate), then* $\sigma_y^2$ *is a monotonic increasing function of* $X$ *for all* $X \geqq 0$.[11]

PROOF:

1. If $f(r) = 0$ for $r \leqq d$ for all $X \geqq 0$, then $\partial\sigma_y^2/\partial X > 0$, since in this case, $\sigma_y^2 = A^2\sigma_r^2$, and $\partial\sigma_y^2/\partial X = 2Ap\sigma_r^2 > 0$.
2. In general, $\sigma_y^2$ is given by

$$\sigma_y^2 = \int_{-1}^{d} m^2 f(r)dr + \int_{d}^{\infty}[Ar - (1-p)X]^2 f(r)dr - (Ey)^2 ,$$

where

$$Ey = -\int_{-1}^{d} mf(r)dr + \int_{d}^{\infty}[Ar - (1-p)X]f(r)dr .$$

Hence,

$$\frac{\partial\sigma_y^2}{\partial X} = 2\int_{d}^{\infty}[Ar - (1-p)X - Ey][pr - (1-p)]f(r)dr .$$

3. Let $g(r) = [Ar - (1-p)X - Ey]$, and let $h(r) = [pr - (1-p)]$. Then $\partial\sigma_y^2/\partial X$ may be rewritten as

$$\frac{\partial\sigma_y^2}{\partial X} = 2\int_{d}^{(1-p)/p} g(r)h(r)f(r)dr + 2\int_{(1-p)/p}^{\infty} g(r)h(r)f(r)dr .$$

We consider, in turn, the signs associated with each of these two integrals:

[10] In this paper, the term "risk" is treated as being synonymous with variance. This interpretation differs from that of Boulding, *op. cit.*, who uses the range of income, and from that of Domar and Musgrave, who interpret risk as the expected value of negative income levels, and contrast risk, as thus defined, with "yield," defined as the expected value of positive income levels. See Evsey D. Domar and R. A. Musgrave, "Proportional Income Taxation and Risk-taking," *Quarterly Journal of Economics*, LVIII (May, 1944), pp. 387–422.

[11] In the proof of this lemma it is assumed that $X$ is a continuous variable. Under this assumption, $\partial\sigma_y^2/\partial X$ exists everywhere except perhaps at the value of $X$ where the probability of default becomes positive. Since $\sigma_y^2$ is continuous in $X$ at this point, it is sufficient to show that $\partial\sigma_y^2/\partial X > 0$ at every point where $\partial\sigma_y^2/\partial X$ exists, in order to prove that $\sigma_y^2$ is a monotonic increasing function of $X$ for all $X \geqq 0$. A similar argument applies for Lemma 2 below.

4. For $d \leqq r \leqq (1-p)/p$, $g(r) \leqq 0$, since $Ey \geqq AEr - (1-p)X \geqq A[(1-p)/p] - (1-p)X \geqq Ar - (1-p)X$ for $r \leqq (1-p)/p$. Further, $h(r) \leqq 0$, since $pr \leqq 1-p$ for $r \leqq (1-p)/p$. Hence,

$$2\int_{d}^{(1-p)/p} g(r)h(r)f(r)dr \geqq 0 .$$

5. Next consider the sign of the second integral:

Let $\alpha = \dfrac{1}{\int_{(1-p)/p}^{\infty} f(r)dr}$. Then, by assumption, $\alpha$ exists, and, by construction, $\alpha \geqq 1$. Then

$$2\int_{(1-p)/p}^{\infty} g(r)h(r)f(r)dr = \frac{2}{\alpha}\int_{(1-p)/p}^{\infty} g(r)h(r)\alpha f(r)dr .$$

Define $f^*(r)$ by

$$f^*(r) = \begin{cases} \alpha f(r) & \text{for } r \geqq \dfrac{1-p}{p} \\ 0 & \text{for } r < \dfrac{1-p}{p} \end{cases} .$$

Then $E^*y = \int_{(1-p)/p}^{\infty} [Ar - (1-p)X]f^*(r)dr \geqq Ey$, since $f^*(r)$ either dominates $f(r)$ or is identical to $f(r)$.[12] Hence

$$\frac{2}{\alpha}\int_{(1-p)/p}^{\infty} [Ar - (1-p)X - Ey][pr - (1-p)]f^*(r)dr$$

$$\geqq \frac{2}{\alpha}\int_{(1-p)/p}^{\infty} [Ar - (1-p)X - E^*y][pr - (1-p)]f^*(r)dr .$$

But the second expression of the inequality equals

$$\frac{1}{\alpha}\frac{\partial\sigma_y^{2*}}{\partial X} = \frac{2Ap\sigma_r^{2*}}{\alpha} > 0 ,$$

since $\sigma_r^{2*} > 0$. Therefore,

$$2\int_{(1-p)/p}^{\infty} g(r)h(r)f(r)dr > 0 \qquad \text{for all } X \geqq 0 .$$

Hence $\partial\sigma_y^2/\partial X > 0$ for all $X \geqq 0$ for which $\partial\sigma_y^2/\partial X$ exists.

LEMMA 2. *The Principle of Increasing Risk—Equity Capital: If a firm operates with a fixed amount of loan capital under the condition of stochastic constant returns, and if it is assumed that* $Er$ *and* $\sigma_r^2$ *exist,* $\sigma_r^2 > 0$, *and* $Er > (1-p)/p$, $(0 < p \leqq 1)$, *then* $\sigma_y^2$ *is a*

[12] The referee has pointed out that a proof of the inequality $E^*y \geqq Ey$ can be constructed without introducing the concept of stochastic dominance.

*monotonic increasing function of $m$ for all $m \geqq 0$.*

PROOF:

1. $\dfrac{\partial \sigma_y^2}{\partial m} = 2\left[\int_{-1}^{a} (m + Ey) f(r)dr + \int_{a}^{\infty} [Ar - (1-p)X - Ey] r f(r)dr\right]$.
2. The first integral of the expression above is obviously non-negative, since $m \geqq 0$, $Ey > 0$, and $f(r) \geqq 0$ for all $r$.
3. Multiplying the second integral by $p$, $0 < p \leqq 1$, gives:

$$p\int_{a}^{\infty} [Ar - (1-p)X - Ey]\left[r - \frac{1-p}{p}\right] f(r)dr + (1-p)\int_{a}^{\infty} [Ar - (1-p)X - Ey] f(r)dr .$$

By Lemma 1, the first integral in the expression above is positive. Further, the second integral is non-negative, since $(1-p) \geqq 0$, and $Ey \leqq \int_{a}^{\infty} [Ar - (1-p)X] f(r)dr$.
Hence $\partial \sigma_y^2/\partial m > 0$ for all $m \geqq 0$ for which $\partial \sigma_y^2/\partial m$ exists.

Lemma 1 leads to the following proposition on risk aversion:

PROPOSITION 1. *Under the assumptions of Lemma* 1, *and given the further assumption that $\partial Eu/\partial Ey > 0$, then a necessary condition for a determinate finite optimal size of firm at a point where $\partial Eu/\partial X$ exists is that the entrepreneur's utility function be characterized by risk aversion, that is, $\partial Eu/\partial \sigma_y^2 < 0$.*

PROOF:

1. If a determinate finite optimal size of firm exists at $X = \bar{X}$, $\bar{X} \geqq 0$, then either $(\partial Eu/\partial X)_{X=\bar{X}} = [(\partial Eu/\partial Ey)(\partial Ey/\partial X) + (\partial Eu/\partial \sigma_y^2)(\partial \sigma_y^2/\partial X)]_{X=\bar{X}} \leqq 0$, or $(\partial Eu/\partial X)_{X=\bar{X}}$ does not exist.
2. If $(\partial Eu/\partial X)_{X=\bar{X}}$ exists, then:
   a) $\partial Eu/\partial Ey > 0$, by assumption.
   b) $\partial \sigma_y^2/\partial X > 0$, by Lemma 1.
3. $\partial Ey/\partial X$ is given by $\partial Ey/\partial X = \int_{a}^{\infty} [pr - (1-p)] f(r)dr$. Therefore $\partial Ey/\partial X > 0$ if and only if

$$\int_{a}^{\infty} r f(r)dr > \frac{1-p}{p}\int_{a}^{\infty} f(r)dr .$$

By assumption,

$$Er = \int_{-1}^{a} r f(r)dr + \int_{a}^{\infty} r f(r)dr > \frac{1-p}{p}\int_{-1}^{a} f(r)dr + \frac{1-p}{p}\int_{a}^{\infty} f(r)dr = \frac{1-p}{p} .$$

But $d = X/(m + pX) - 1 \leqq (1/p) - 1$, for all $X \geqq 0$. For $X = 0$, $d = -1$, and for $X > 0$, $(m + pX)/X = m/X + p \geqq p$.
Hence $\int_{-1}^{d} rf(r)dr \leqq (1-p)/p \int_{-1}^{d} f(r)dr$. Since $Er > (1-p)/p$, this implies that $\int_{d}^{\infty} rf(r)dr > (1-p)/p \int_{d}^{\infty} f(r)dr$, so that $\partial Ey/\partial X > 0$.
Hence $\partial Eu/\partial \sigma_y^2 < 0$ if a determinate finite optimal size of firm exists at a point where $\partial Eu/\partial X$ exists.

It is next shown that Proposition 1 is not vacuous, i.e., that there exist situations in which determinate finite optimal sized firms will exist, given the assumptions underlying Proposition 1. The problem of vacuousness arises because it can be shown that firms are of infinite size if the optimal amount of equity capital invested in a firm is zero.

Let $G(y; X)$ denote the distribution function of $y$. Then, for $m = 0$, $G(y; X)$ is given by

$$G(\bar{y}; X) = \begin{cases} \int_{-1}^{(1-p)/p} f(r)dr, & \text{for } \bar{y} = 0 \\ \int_{-1}^{\bar{r}} f(r)dr, & \text{for } \bar{y} > 0, \text{ where } \bar{r} = \dfrac{\bar{y} + (1-p)X}{pX}. \end{cases}$$

Consider G$(y; X)$ for $m = 0$:
$G(y; X'')$ dominates $G(y; X')$ for all $X'' > X' \geqq 0$, since $G(0; X'') = G(0; X') = \int_{-1}^{(1-p)/p} f(r)dr$, and $G(\bar{y}; X'') = \int_{-1}^{\bar{r}(X'')} f(r)dr < \int_{-1}^{\bar{r}(X')} f(r)dr = G(\bar{y}; X')$, for all $\bar{y} > 0$, since $\partial \bar{r}/\partial X = -\bar{y}p/(pX)^2 < 0$ for all $\bar{y} > 0$.
Since, for $m = 0$, the distribution of income for $X = X''$ dominates the distribution of income for $X = X'$ for all $X'' > X' \geqq 0$, the optimal size of firm is infinite. Thus Proposition 1 is vacuous if the optimal amount of equity capital invested in all firms is zero, given the assumption of a perfect loan market.

An interesting corollary to the statement above is that for $m > 0$ dominance does not arise for all values of $X$, and it is further possible to construct an example in which a member of the "efficient set" of investments for an entrepreneur contains firms with positive amounts of equity capital.[13]

We consider the special case where the entrepreneur either invests his assets in the equity of a particular firm or, alternatively, invests in some risk-free security offering with certainty a payoff of $i$ percent

[13] The term "efficient set" is interpreted in the sense of Markowitz, that is, a set of alternatives such that $\sigma_y^2$ is minimized for given $Ey$, for each value of $Ey$. See Harry Markowitz, *op. cit.*

per dollar invested per period. It is assumed that the subjective probability distribution over rates of return earned by firms is characterized by statistical independence, so that the variance of income for the entrepreneur simply equals the variance of income from his investment in the firm. For given $Ey = Ey_0$, we investigate the change in $\sigma_y^2$ that occurs as funds are shifted out of the risk-free security into equity of the firm. In effect, what is being measured is the effect on variance of replacing loan capital in the firm with equity capital in such a way that $Ey$ remains constant, where the opportunity cost per dollar of equity invested in the firm (in $Ey$ terms) is $i$ dollars. Then,

$$dEy_0 = 0 = \frac{\partial Ey}{\partial X}\,dX + \left(\frac{\partial Ey}{\partial m} - i\right)dm\,;$$

hence

$$dX = -\frac{\left(\frac{\partial Ey}{\partial m} - i\right)}{\frac{\partial Ey}{\partial X}}\,dm$$

$$= -\left[\frac{\int_a^{\bar{r}} rf(r)dr - \int_{-1}^{a} f(r)dr - i}{p\int_a^{\bar{r}} rf(r)dr - (1-p)\int_a^{\bar{r}} f(r)dr}\right]dm = \psi(X, m, i)dm\,.^{14}$$

Further,

$$d\sigma_y^2 = \frac{\partial \sigma_y^2}{\partial X}\,dX + \frac{\partial \sigma_y^2}{\partial m}\,dm$$

$$= \Big\{[p\psi(X, m, i) + 1]\int_a^{\bar{r}} [Ar - (1-p)X - Ey]rf(r)dr$$

$$-(1-p)\psi(X, m, i)\int_a^{\bar{r}} [Ar - (1-p)X - Ey]f(r)dr\Big\}dm\,.$$

It is difficult to give a simple interpretation of the conditions under which $d\sigma_y^2 < 0$ when default risk is present, but if it is assumed that the firm is default risk-free, $\psi(X, m, i) = -(Er - i)/[pEr - (1-p)]$ and $d\sigma_y^2 = A\sigma_r^2[\{-(1-p) + pi\}/\{pEr - (1-p)\}]dm < 0$ for $i < (1-p)/p$, that is, when the rate of return offered on the risk-free investment

[14] It might be noted that the sign of $\partial Ey/\partial m$ is not necessarily positive for all $m \gtrless 0$ under the assumption that $Er > (1-p)/p$. In general, $Ey$ is a decreasing function of $m$ (for $X$ fixed) up to that value of $m = \bar{m}$ such that $\int_{d(\bar{m})}^{\bar{r}} rf(r)dr = \int_{-1}^{d(\bar{m})} f(r)dr$ (if $\bar{m} \neq 0$), and is an increasing function of $m$ beyond that point. In the expression for $dX$, obviously $m \gtrless \bar{m}$.

is less than the interest rate charged on loans to the firm. It might be argued that the market mechanism would insure that $i$ would equal $(1 - p)/p$ if the firm is default risk-free. This result occurs if the beliefs of lenders and entrepreneurs concerning the occurrence of rates of return for firms are identical, but if differences in beliefs are admitted as a possibility, there is no reason to assume this result is inevitable. Thus Proposition 1 is shown to be non-vacuous.

The first section of this paper has dealt with the relationship of determinacy of size of firm to risk aversion under the assumption that loan markets are perfect; in Part 2 below, the relationship between default risk and loan market imperfections limiting size of firm is examined.

## 2. LENDER BEHAVIOR UNDER UNCERTAINTY

The notation of Part 1 is used in this section as well. In addition, we introduce the following set of symbols:

Let

$X_i^j$ = the number of one dollar notes sold by the $i$th firm to the $j$th lender, $j = 1, 2, \cdots, m$,[15]

$\alpha^j \equiv \sum_{i=1}^{n+1} p_i X_i^j$ = the assets held by the $j$th lender during the period, where the $(n + 1)$st borrower is the lender himself (holding of cash), so that $p_{n+1} \equiv 1$,

$p \equiv (p_1, p_2, \cdots, p_n, 1)$ = the vector of prices of notes,

$X^j \equiv (X_1^j, X_2^j, \cdots, X_n^j, X_{n+1}^j)$ = the vector of note holdings by the $j$th lender,

$Z^j \equiv (Z_1^j, Z_2^j, \cdots, Z_n^j, Z_{n+1}^j)$ = the vector of note holdings by "all other lenders," from the $j$th lender's point of view, where $Z_i^j = \sum_{k \neq j} X_i^k$,

$y^j \equiv \sum_{i=1}^{n+1} (R_i - p_i) X_i^j$ = the income of the $j$th lender for the period, where

$$R_i = \begin{cases} 1 & \text{for } r_i \geqq d_i \\ \dfrac{A_i(1 + r_i)}{X_i^j + Z_i^j} \equiv \dfrac{B_i}{X_i} & \text{for } r_i \leqq d_i \end{cases} = \text{the amount repaid per one dollar note issued by the } i\text{th firm},$$

where

$d_i \equiv (X_i/A_i) - 1 \equiv [(X_i^j + Z_i^j)/A_i] - 1$ = the default rate of return for the $i$th firm,

$r \equiv (r_1, r_2, \cdots, r_n, r_{n+1})$ = the vector of rates of return of firms,

[15] Thus $X_i \equiv \sum_{j=1}^{m} X_i^j$ = the number of one dollar notes sold by the $i$th firm to all lenders.

$A \equiv (A_1, A_2, \cdots, A_n, A_{n+1})$ = the vector of asset holdings at the beginning of the period for firms.[18]

Let $f^j(r; A)$ denote the subjective probability distribution held by the $j$th lender with respect to rates of return earned by the borrowing firms. $u^j = u^j(y^j)$ is the measurable utility function of the $j$th lender, and $Eu^j = \psi^j(Ey^j, \sigma^2_{y^j})$.

It is apparent from the expression for $y^j$ given above that the income payoff to the $j$th lender from notes purchased from the $i$th borrower depends not only upon the $j$th lender's actions, but also upon the actions of other lenders. If the loan market is viewed in the Walrasian sense, where a "referee" announces prices of notes and adjusts the price vector according to excess demand conditions until markets are cleared, so that participants in the market are cognizant only of price data, the $j$th lender operates under two kinds of uncertainty: (i) uncertainty as to the the value that $r$ will take on, and (ii) uncertainty as to the value taken on by $Z^j$. We consider only two of the many possible cases for analysis: (1) the case where $Z^j = Z^j_0$, that is, $Z^j$ is assumed to be known with certainty, and (2) the case where the $j$th lender is "ignorant" of $Z^j$, that is, the case in which there does not exist a subjective probability distribution to order the beliefs of the $j$th lender with respect to portfolio choices of other lenders. In each case, the demand functions for notes are examined under the simplifying assumptions of statistical independence and stochastic constant returns. It is further assumed that no second hand markets for notes exist, and that lenders are subject to no liquidity constraints.

*Case* 1. $Z^j = Z^j_0$.

When the $j$th lender is in a state of certainty concerning the actions of other lenders, the rule of rational conduct for the lender is to choose that portfolio of notes, $X^j$, that maximizes expected utility, subject to non-negativity conditions and the budget constraint, that is,

$$\max Eu^j$$

$$\text{subject to: (i)} \quad \sum_{i=1}^{n+1} p_i X^j_i = \alpha^j,$$

$$\text{(ii)} \quad X^j_i \geqq 0, \text{ for all } i.$$

[18] For symmetry, the $(n+1)$st borrower (holding of cash) has been treated the same as other borrowers throughout the notation above. The $(n+1)$st borrower should be viewed as one for which $p_{n+1} \equiv 1$, $m_{n+1} \equiv 0$, so that $A_{n+1} \equiv X_{n+1}$; $r_{n+1} \equiv 0$, for all values of the $r_i$'s, $i \neq n+1$. Thus $d_{n+1} \equiv 0$, and $R_{n+1} = 1$. Implicit in this treatment of cash holdings is the assumption that lenders are concerned only with money income and not with real income in making their portfolio choices.

Let $L$ denote the Lagrange expression. Then[17]

$$L = Eu + \lambda(\alpha - \sum_i p_i X_i),$$

and the necessary conditions for a maximum of $Eu$ with respect to $X$ at a point $\bar{X} = (\bar{X}_1, \bar{X}_2, \cdots, \bar{X}_n, \bar{X}_{n+1})$ where $\partial Eu/\partial X_i$ exists for all $i$ are given by[18]

(i) $\bar{X}_i \left(\frac{\partial L}{\partial X_i}\right)_{x=\bar{x}} = 0$ for all $i \Rightarrow$

$$\bar{X}_i \left(\frac{\partial Eu}{\partial Ey}\frac{\partial Ey}{\partial X_i} + \frac{\partial Eu}{\partial \sigma_y^2}\frac{\partial \sigma_y^2}{\partial X_i} - \lambda p_i\right)_{x=\bar{x}} = 0 \text{ for all } i;$$

(ii) $\sum_i p_i \bar{X}_i = \alpha, \qquad \lambda \geqq 0;$

(iii) $\bar{X}_i \geqq 0, \left(\frac{\partial L}{\partial X_i}\right)_{x=\bar{x}} \leqq 0$ for all $i$.

Under stochastic constant returns and independence, the expressions for $\partial Ey/\partial X_i$ and $\partial \sigma_y^2/\partial X_i$ reduce to

$$\frac{\partial Ey}{\partial X_i} = X_i \frac{\partial ER_i}{\partial X_i} + ER_i - p_i,$$

and

$$\frac{\partial \sigma_y^2}{\partial X_i} = 2X_i \sigma_{R_i}^2 + X_i^2 \frac{\partial \sigma_{R_i}^2}{\partial X_i},$$

where

$$ER_i = \int_{-1}^{d_i} \left[\frac{A_i(1+r_i)}{X_i + Z_{i_0}}\right] f(r_i)dr_i + \int_{d_i}^{\infty} f(r_i)dr_i$$

= the expected repayment per one dollar note issued by the $i$th firm,

and

$$\sigma_{R_i}^2 = \int_{-1}^{d_i} \left[\frac{A_i(1+r_i)}{X_i + Z_{i_0}}\right]^2 f(r_i)dr_i + \int_{d_i}^{\infty} f(r_i)dr_i - (ER_i)^2$$

= the variance of repayments per one dollar note issued by the $i$th firm.

$\partial Ey/\partial X_i$ will be termed the "marginal expected profitability" of loans to the $i$th firm. Then $\partial Ey/\partial X_i \leqq 0$ for

$$\frac{m_i X_i}{(X_i + Z_{i_0})^2} \int_{-1}^{d_i} (1 + r_i) f(r_i)dr_i \gtrsim ER_i - p_i,$$

under the assumption that $ER_i > p_i$. Further, under independence and stochastic constant returns, diminishing marginal expected profitability

[17] Since all of the variables in this section refer to the $j$th lender, the superscript "$j$" is omitted from all variables.

[18] $p$, the vector of note prices, is taken as a parameter of the lender's decision problem.

of loans characterizes note purchases by lenders. Assume $ER_i > p_i$, and $\int_{-1}^{d_i} f(r_i)dr_i > 0$, that is, the probability of default is positive. Then $\partial^2 Ey/\partial X_i^2 = X_i(\partial^2 ER_i/\partial X_i^2) + 2(\partial ER_i/\partial X_i)$ exists for all $X_i \geqq 0$, when $X_i$ is treated as a continuous variable and $f(r_i)$ is assumed continuous. Under stochastic constant returns,

$$\frac{\partial^2 Ey}{\partial X_i^2} = X_i\left[\frac{-m_i^2}{(X_i + Z_{i_0})^2 A_i^2}(1 + d_i)f(d_i) + \frac{2m_i}{(X_i + Z_{i_0})^2}\int_{-1}^{d_i}(1 + r_i)f(r_i)dr_i\right]$$

$$- \frac{2m_i}{(X_i + Z_{i_0})^2}\int_{-1}^{d_i}(1 + r_i)f(r_i)dr_i < 0 \qquad \text{by inspection.}$$

The principle of diminishing marginal expected profitability of loans follows from the assumptions of statistical independence and stochastic constant returns, because an increase in the number of notes purchased from a borrower leads to an increase in the default rate of return, that is, $\partial d_i/\partial X_i = m_i/A_i^2 > 0$. Since the lender is limited in any case to a certain maximum payoff (the face value of the note), the effect of this shift in the size of the default rate of return is to make the lender less and less willing to purchase additional notes as the number purchased increases. As a consequence, diversification of note holdings by expected income maximizing lenders may be rationalized.

PROPOSITION 2. *Risk aversion is not a necessary condition for diversification of note holdings.*

A simple illustration of the proposition is provided by a case in which an expected income maximizing lender either holds cash or notes issued by a firm. If the assets held by the lender are sufficiently large, and if the subjective probability distribution is properly chosen, it is apparent that the principle of diminishing marginal expected profitability of loans will insure that a point will be reached at which the marginal expected profitability from note purchases will become negative, thus insuring diversification of holdings between cash and notes.

The underlying reason for diversification of note holdings by expected-income-maximizing lenders lies in the fact that the expected value of income for a lender is a function of variance of income for the borrower, so that an increase in note purchases which increases the variance of income for the borrower (see Lemma 1 above) reduces expected repayments per note to the lender. A conclusion of some interest follows from this: Borrowers who are expected income maximizers might find it to their advantage to practice diversification in the operations of their firms and thus reduce the variance of income

even if they deal with expected income maximizing lenders; such diversification might more than pay for itself (in expected income terms) through the additional loan capital that would be made available by lenders, or through the decrease in cost of a fixed amount of loan funds. Diversification on the part of both borrowers and lenders may thus be explained even if risk aversion (in the sense of this paper) is absent.

In a manner identical to that employed above to determine $\partial Ey/\partial X_i$, the relationship between note purchases and variance of the lender's income may be determined:

$$\frac{\partial \sigma_y^2}{\partial X_i} = 2X_i\sigma_{R_i}^2 + X_i^2\frac{\partial \sigma_{R_i}^2}{\partial X_i}$$

$$= 2X_i\left[\sigma_{R_i}^2\left(1 - \frac{X_i m_i}{A_i(X_i + Z_{i_0})}\right) + \frac{m_i X_i}{A_i(X_i + Z_{i_0})}(1 - ER_i)\int_{a_i}^{\bar{r}} f(r_i)dr_i\right] \geq 0 .$$

Since variance of income is an increasing function of the number of notes purchased, risk-averting lenders, as well as expected-income maximizing lenders, will limit the number of notes purchased from a given borrower, when independence and stochastic constant returns are assumed, and $Z^j$ is known with certainty.

Under these assumptions, it can be shown that the demand curve for notes issued by any firm is downward sloping, because the distribution of income for $p_i = p_i'$ dominates the distribution of income for $p_i = p_i''$ for all $p_i' < p_i''$, if $X_i$ is assumed fixed:

Let $G(y_i; p_i)$ denote the distribution function over income for the lender. Then $G(y_i; p_i)$ is given by

$$G(\bar{y}_i; p_i) = \begin{cases} 1 & \text{for } \bar{y}_i = (1 - p_i)X_i \\ \int_{-1}^{\bar{r}_i} f(r_i)dr_i & \text{for } \bar{y}_i < (1 - p_i)X_i , \end{cases}$$

where

$$\bar{r}_i = \left[\frac{\bar{y}_i}{X_i} + p_i\right]\left[\frac{X_i + Z_{i_0}}{A_i}\right] - 1 .$$

Then $G(y_i; p_i')$ dominates $G(y_i; p_i'')$ for all $p_i' < p_i''$, since

$$\frac{\partial \bar{r}_i}{\partial p_i} = \left[\frac{X_i + Z_{i_0}}{A_i}\right]\left[1 - \left(\frac{\bar{y}_i + p_i X_i}{A_i}\right)\left(\frac{X_i + Z_{i_0}}{A_i}\right)\right] > 0 .$$

Of particular interest is the relationship between default risk and what may be referred to as "capital rationing," that is, the inability

of firms to obtain additional loan capital through a lowering of the price of notes.[19] While no general results are presented here, for at least one special case the inevitability of capital rationing with respect to the $k$th borrower may be established.

Assume $Z_k = 0$, and $p_k$ is so small that default is certain. Let $p_k^*$ be the largest value of $p_k$ for which $\int_{-1}^{d_k} f(r_k)dr_k = 1$ for $X_k = X_k^*$, and $Z_k = 0$. Then, for $p_k < p_k^*$, $p_k X_k = p_k^* X_k^*$,

$$ER_k = \int_{-1}^{d_k} \frac{A_k(1 + r_k)}{X_k} f(r_k)dr_k ,$$

and

$$\sigma_{R_k}^2 = \int_{-1}^{d_k} \left[\frac{A_k(1 + r_k)}{X_k}\right]^2 f(r_k)dr_k - (ER_k)^2 .$$

Therefore,

$$\frac{dEy_k}{dX_k} = \frac{X_k}{A_k} f(d_k) = 0 .$$

Also,

$$\frac{d\sigma_y^2}{dX_k} = X_k^2 \frac{\partial \sigma_{R_k}^2}{\partial X_k} + 2X_k \sigma_{R_k}^2 = \frac{X_k^2}{A_k}(1 - 2ER_k)f(d_k) = 0 ,$$

since

$$f(r_k) = 0 \qquad \text{for } r_k > \frac{X_k^*}{m_k + p_k^* X_k^*} - 1 = d_k^* ;$$

and

$$\frac{dd_k}{dX_k} > 0 \qquad \text{for } p_k X_k = p_k^* X_k^* .$$

Since the expected value and variance of income remain constant for a constant amount lent to a firm, when the price of notes has fallen so low that default is certain, the lender is indifferent among all alternatives involving the same total investment in notes of the firm for $p_k \leqq p_k^*$, that is, the demand curve is of unit elasticity below this price. This conclusion applies both to expected income maximizing and risk averting lenders. Hence:

PROPOSITION 3. *Risk aversion is not a necessary condition for the existence of capital rationing.*

From the analysis above we may make a (highly tentative) suggestion

[19] In the context of the model under consideration, "capital rationing" on the loan market is said to occur whenever the demand curve for notes issued by a firm is inelastic or of unit elasticity.

as to the importance of capital rationing in the loan markets: Lenders act as if they believed default were certain. Such an assumption, coupled with the others underlying the model of this paper, is sufficient to explain the inability of firms to obtain additional funds by agreeing to an increase in interest rates.

*Case 2. $Z^j$ unknown.*

We next consider the problem of portfolio choices by a lender in a state of ignorance concerning the actions of other lenders, although it is assumed that a subjective probability distribution characterizes the lender's beliefs with respect to the occurrence of rates of return for firms. While many different criteria have been suggested as rules of "rational" conduct in this situation, the analysis here is restricted to a modified form of the minimax rule, that of maximizing the minimum expected utility from portfolio choices.[20]

Formally, the problem may be stated as follows:

$$\max_{X} \min_{Z} Eu(X, Z) ,$$

$$\text{subject to: (i)} \quad X_i \geqq 0 \text{ for all } i ,$$

$$\text{(ii)} \quad Z_i \geqq 0 \text{ for all } i ,$$

$$\text{(iii)} \quad \sum_{i=1}^{n+1} p_i X_i = \alpha ,$$

$$\text{(iv)} \quad \sum_{i=1}^{n+1} p_i Z_i = \sum_{s \neq j} \alpha^s .$$

The solution to this problem may be obtained by first determining a value of $Z$, denoted by $Z^*$, that minimizes $Eu$, subject to the restrictions noted. Then $Eu(X, Z^*)$ is maximized with respect to $X$, subject to the restrictions on $X$. The optimal portfolio is thus one that maximizes the expected utility of the $j$th lender for the least desirable portfolio of note holdings by other lenders.

Under the assumptions of stochastic constant returns and independence, $Ey$ is a decreasing function of $Z$ (for $X$ fixed), while $\sigma_y^2$ first increases and then decreases as $Z$ increases.[21] It can be shown,

[20] For a discussion of the applications of this rule in games against nature, see Leonid Hurwicz, "Some Specification Problems and Applications to Econometric Models" (abstract), *Econometrica*, XIX (July, 1951), pp. 333-334.

[21] The expressions for $\partial Ey/\partial Z_i$ and $\partial \sigma_y^2/\partial Z_i$ are:

$$\frac{\partial Ey}{\partial Z_i} = \frac{-m_i X_i}{(X_i + Z_i)^2} \int_{-1}^{a_i} (1 + r_i) dr_i \leqq 0 \qquad \text{for all } X_i, Z_i \geqq 0 ,$$

$$\frac{\partial \sigma_y^2}{\partial Z_i} = \frac{-2 m_i X_i^2}{(X_i + Z_i) A_i} \left[ \sigma_{R_i}^2 - (1 - ER_i) \int_{a_i}^{\infty} f(r_i) dr_i \right] \gtrless 0 .$$

however, that the distribution of income for the $j$th lender from notes purchased from the $i$th borrower when $Z_i = Z_i''$ is stochastically dominated by the distribution for $Z_i = Z_i'$ for $Z_i' < Z_i''$, for all $X_i \geqq 0$:

Let $y_i^j$ denote the income received by the $j$th lender from notes purchased from the $i$th borrower. Then $y_i^j$ is given by

$$y_i^j = (R_i - p_i)X_i ,$$

where

$$R_i = \begin{cases} 1 & \text{for } r_i \geqq d_i \equiv \dfrac{X_i + Z_i}{A_i} - 1 \\ \dfrac{A_i(1 + r_i)}{X_i + Z_i} & \text{for } r_i \leqq d_i ; \end{cases}$$

so that

$$y_i^j = \begin{cases} (1 - p_i)X_i & \text{for } r_i \geqq d_i \\ \left(\dfrac{A_i(1 + r_i)}{X_i + Z_i} - p_i\right)X_i & \text{for } r_i \leqq d_i . \end{cases}$$

Furthermore, if $G(y_i^j; Z_i)$ is the distribution function of $y_i^j$,

$$G(\bar{y}_i^j; Z_i) = \begin{cases} 1 & \text{for } \bar{y}_i^j = (1 - p_i)X_i \\ \int_{-1}^{\bar{r}_i} f(r_i)dr_i & \text{for } \bar{y}_i^j < (1 - p_i)X_i , \end{cases}$$

where

$$\bar{r}_i = \left(\frac{\bar{y}_i^j}{X_i} + p_i\right)\left(\frac{X_i + Z_i}{A_i}\right) - 1 ,$$

so that

$$\frac{\partial \bar{r}_i}{\partial Z_i} = \left(\frac{\bar{y}_i^j}{X_i} + p_i\right)\frac{m_i}{A_i^2} \geqq 0 \qquad \text{for all } \bar{y}_i^j \geqq -p_i X_i .$$

Hence $G(y_i^j; Z_i')$ dominates $G(y_i^j; Z_i'')$ for all $Z_i'' > Z_i' \geqq 0$.

Thus, for any borrower, the least desirable holding of notes by all other lenders occurs (from the $j$th lender's point of view) when these note holdings are as large as possible.

If it is now assumed that the new loan market is characterized by perfect competition, in the sense that the amount of funds possessed by the $j$th lender, $\alpha$, is so small relative to the amount of funds possessed by all other lenders, $\sum_{k \neq j} \alpha^k$, that the $j$th lender acts as if these other lenders possessed an unlimited amount of funds, then the optimal portfolio of notes for the $j$th lender is one that maximizes $Eu$ considered as a limiting expression as $Z_i$ approaches infinity for all $i$.

Let $\widetilde{ER}_i$ denote $\lim_{x_i \to \infty} ER_i$, and $\tilde{\sigma}^2_{R_i}$ denote $\lim_{x_i \to \infty} \sigma^2_{R_i}$. The expressions for $\widetilde{ER}_i$ and $\tilde{\sigma}^2_{R_i}$ are given by

$$\widetilde{ER}_i = p_i \int_{-1}^{e_i} (1 + r_i) f(r_i) dr_i + \int_{-1}^{\infty} f(r_i) dr_i ,$$

and

$$\tilde{\sigma}^2_{R_i} = p_i^2 \int_{-1}^{e_i} (1 + r_i)^2 f(r_i) dr_i + \int_{e_i}^{\infty} f(r_i) dr_i - (\widetilde{ER}_i)^2 ,$$

where

$$e_i \equiv \lim_{x_i \to \infty} d_i = \frac{1 - p_i}{p_i} .$$

Therefore, $\lim_{x \to \infty} Eu(X, Z)$, a maximum with respect to $X$ at a point $\bar{X} = (\bar{X}_1, \bar{X}_2, \cdots, \bar{X}_{n+1})$, implies:

(i) $\bar{X}_i \left( \dfrac{\partial Eu}{\partial Ey} (\widetilde{ER}_i - p_i) + \dfrac{\partial Eu}{\partial \sigma_y^2} 2\bar{X}_i \tilde{\sigma}^2_{R_i} - \lambda p_i \right) = 0$ for all $i$,

(ii) $\bar{X}_i \geqq 0$ for all $i$, $\lambda \geqq 0$,

(iii) $\sum_{i=1}^{n+1} p_i \bar{X}_i = \alpha$,

so that, for $\bar{X}_i > 0$,

$$\frac{\dfrac{\partial Eu}{\partial Ey} (\widetilde{ER}_i - p_i) + \dfrac{\partial Eu}{\partial \sigma_y^2} 2\bar{X}_i \tilde{\sigma}^2_{R_i}}{p_i} = \lambda .$$

As is indicated by the above process, $\widetilde{ER}_i$ and $\tilde{\sigma}^2_{R_i}$ are independent of $X_i$ and depend instead only on the lender's subjective probability distribution and $p_i$. Thus, for a lender with a linear utility function, the minimaxing portfolio consists entirely of notes issued by that borrower for which $\widetilde{ER}_i - p_i$ is a maximum (or any probability distribution over notes with equal (maximum) values). Diversification of note holdings thus implies risk aversion, if the marginal expected utility of expected income is positive.

If cash is held, $\lambda = 0$, since $\widetilde{ER}_{n+1} = p_{n+1} \equiv 1$, and $\tilde{\sigma}^2_{R_{n+1}} = 0$. Since cash is always a possible alternative to note purchases, minimaxing lenders will hold only notes issued by firms for which $\widetilde{ER}_i - p_i \geqq 0$.

In the same manner as that employed in Case 1 above, it can be shown that demand curves for notes are downward sloping, and that capital rationing is inevitable for both expected income-maximizing

and risk averting lenders once the price of notes has fallen so low that default is certain. As might be expected, the critical price of notes at which capital rationing occurs is higher in the ignorance case than when $Z^1$ is known with certainty.

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# ON A THEOREM OF RELATIVE STABILITY*

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ALTHOUGH MANY writers have discussed the necessary and sufficient conditions for the relative stability of balanced growth in Leontief's dynamic system, these conditions have not yet been given in matrix form. The following theorem may, therefore, be of some interest.

Let $X_t = AX_t + B(X_{t+1} - X_t)$ represent a closed, dynamic, input-output system, where $X_t$ is an $n$-dimensional vector of output levels in period $t$, $A$ an $n \times n$ input-output matrix, and $B$ an $n \times n$ stock-flow matrix. Assume that $B$ is of rank $n$, and write $C = B^{-1}(I - A)$. The system of difference equations can then be written in matrix form:

$$X_{t+1} = (I + C)X_t \,. \tag{1}$$

Professors Solow and Samuelson [2] have proved that if the functions $H^i(X_1, X_2, \cdots, X_n)$ are homogeneous of the first degree and have positive partial derivatives everywhere, the system of nonlinear difference equations $X_i(t+1) = H^i[X_1(t), \cdots, X_n(t)]$, $(i=1, \cdots, n)$, has a unique balanced growth solution which is relatively stable. If $I + C > 0$, equation (1) is reduced to a special case of the Solow-Samuelson system. In this paper we shall assume only $(I - A)^{-1} > 0$, which does not necessarily imply $(I + C) > 0$. We shall prove.

THEOREM. *If* $(I - A)^{-1} > 0$, *then* (1) *has a balanced growth solution uniquely determined by the initial conditions. The balanced growth is relatively stable (in the Solow-Samuelson sense* [2]*) if, and only if, there exists a (finite) positive integer* $m$ *such that* $(I + C)^m > 0$.

PROOF: Let $(\lambda_1, \cdots, \lambda_n)$, $(\mu_1, \cdots, \mu_n)$ and $(\nu_1, \cdots, \nu_n)$ be characteristic roots of the matrices $I + C$, $(I + C)^m$, and $C^{-1}$, respectively. Let $\lambda_i S_i = (I + C)S_i$, $\mu_i T_i = (I + C)^m T_i$ and $\nu_i U_i = C^{-1}U_i$, $(i=1, \cdots, n)$. We find that $\lambda_i^m S_i = (I + C)^m S_i$ and $(\lambda_i - 1)S_i = CS_i$, so that

$$\lambda_i^m = \mu_i \quad \text{and} \quad \lambda_i = 1 + \nu_i^{-1} \qquad (i = 1, \cdots, n), \tag{2}$$

and

$$S_i = T_i = U_i \qquad (i = 1, \cdots, n). \tag{3}$$

Since $(I - A)^{-1} > 0$, and $B$ is a non-negative matrix of rank $n$, $C^{-1}$ is a positive matrix. It has, therefore, a unique, positive, characteristic root,

* Manuscript received July 26, 1960, revised October 11, 1960.

say $\nu_1$, with which a positive eigen-vector $U_1$ can be associated[1]. From equations (2) and (3) we find that $\lambda_1 > 0$ and $S_1 > 0$.

The general solution to equation (1) is given in the form[2]

$$X_t = h_1\lambda_1^t S_1 + \cdots\cdots\cdots + h_n\lambda_n^t S_n ,$$

where $h_1, \cdots, h_n$ are determined by the initial conditions. The particular solution $\lambda_1^t S_1$ is referred to as a balanced growth solution; it is relatively stable if, and only if, $\lambda_1 > |\lambda_i|$, $(i=2, \cdots, n)$.

*Sufficiency.* From (2) and (3), we find that $\mu_1 > 0$ and $T_1 > 0$. Since $(I + C)^m > 0$, $\mu_1$ is the Frobenius root which is greater than any other root, i. e., $\mu_1 > |\mu_i|$, $(i=2, \cdots, n)$[3]. This implies that $\lambda_1 > |\lambda_i|$, $(i = 2, \cdots, n)$.

*Necessity.* Let $V_1\nu_1 = V_1C^{-1}$; then $V_1 > 0$. Clearly, $\lambda_i V_1 S_i = V_1(I + C)S_i = (1 + \nu_1^{-1})V_1 S_i = \lambda_1 V_1 S_i$, so that

$$V_1 S_i = 0 \qquad (i = 2, \cdots, n), \tag{4}$$

because $\lambda_1 > |\lambda_i|$. Let $E_i$ be a column vector which has 1 for the $i$th component and zeros for all other components. A path starting from $E_i$ must satisfy

$$E_i = h_1 S_1 + \cdots\cdots\cdots + h_n S_n , \tag{5}$$

and

$$(I + C)^t E_i = h_1\lambda_1^t S_1 + \cdots\cdots\cdots + h_n\lambda_n^t S_n \quad (t = 1, 2, \cdots). \tag{6}$$

Taking (4) into account, we find from (5) that $h_1 = V_1E_i / V_1S_1 > 0$. Since $\lambda_1 > |\lambda_i|$, $(i = 2, \cdots, n)$, there exists a (finite) positive integer $k_i$ such that (6) is positive for all $t \geqq k_i$. Let $m_i$ be the smallest value of such a $k_i$. If we define $m$ as max $(m_1, m_2, \cdots, m_n)$, then $(I + C)^m(E_1, E_2, \cdots, E_n) = (I + C)^m > 0$.

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[1] DEBREU, G. AND I. N. HERSTEIN, "Nonnegative Square Matrices," *Econometrica*, XXI (October, 1953), 597-607.

[2] SOLOW, R. M. AND P. A. SAMUELSON, "Balanced Growth under Constant Returns to Scale," *Econometrica*, XXI (July, 1953), 412-424.

[1] See [1]. The Property of uniqueness has been proved in [2].

[2] For simplicity's sake we assume that $\lambda_1, \cdots, \lambda_n$ are distinct. Even if there are some multiple roots, the following argument still holds, *mutatis mutandis*.

[3] See [1].

INTERNATIONAL ECONOMIC REVIEW
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# PROFESSOR DEBREU ON "THEORY OF VALUE": A REVIEW ARTICLE*

BY MITSUO SAITO

## 1. INTRODUCTION

MANY IMPORTANT contributions have recently been made by Professors K. J. Arrow, G. Debreu, and others to existence and optimality problems of competitive equilibrium of the Walrasian type. This volume,[1] collecting the chief results of the author's various articles into a systematic whole will, in this reviewer's opinion, acquire a position in the history of general equilibrium theory comparable to that of Professor Hicks' classical book,[2] although the former will probably fail in being read as widely as the latter owing to its highly mathematical character.

In this book with the subtitle *An Axiomatic Analysis of Economic Equilibrium*, the Walrasian theory of general equilibrium is reduced to a set of axioms which are assumed to hold for some entities named commodities, prices, production sets and so forth. The interpretations of the theory are not necessarily unique; any results which can be derived from the axioms are valid for any concrete examples of the abstract model. Thus "the theory, in the strict sense, is logically entirely disconnected from its interpretations".[3]

As regards the theoretical aspects of the book, the reviewer has no comment other than the following eulogistic remarks. First, several remarkable improvements are made in the discussion of the existence of equilibrium (Chapter 5). In all the existing proofs for this problem, one indispensable, troublesome step is to chase equilibrium solutions into a suitable bounded region. In this volume, however, the corresponding discussion is made much more systematically with the aid of the concepts of "positive semi-independency" and "asymptotic cone". Moreover, some of the assumptions under which the existence of competitive equilibrium has been hitherto proved are relaxed. For example, convexity is no longer assumed on the individual production sets, but only on the total production sets.[4] The complication added by this

* Manuscript received January 30, 1961, revised February 28, 1961.

[1] G. Debreu, *Theory of Value*, Cowles Foundation Monograph No. 17 (New York: John Wiley & Sons, 1959).

[2] J. R. Hicks, *Value and Capital* (Oxford: Clarendon Press, 1939).

[3] G. Debreu, *op. cit.*, p. viii.

[4] Debreu owes this idea to Professor Uzawa.

relaxation is successfully met by the author through a careful and ingenious modification of the argument for the original non-relaxed case. Finally, prices are so broadly interpreted in the beginning that negative prices are allowed; they are gradually specialized as certain assumptions on production sets and/or preferences are introduced, until they coincide with ordinary non-negative prices. This procedure will throw light on an aspect of the concept of price.

## 2. DEBREU'S INTERPRETATION OF THE THEORY

Debreu gives the theory of competitive equilibrium a quasi-dynamic interpretation which may be summarized as follows: (1) The economy consists of producers and consumers; every producer knows his future production possibilities, and every consumer knows his future consumption possibilities and his future resources.[5] (2) The interval of time over which economic activities take place is divided into a finite number of elementary periods of equal length. (3) There exists no money in the economy. (4) A commodity is characterized by its physical properties, the date at which it will be available, and the location at which it will be available; two commodities of the same physical properties are treated as different commodities if they are available at different periods or at different places. (5) Given the price system, each producer maximizes his profit subject to his production possibilities; given the price system and the wealth constraint, each consumer chooses a budget to which no other budget in his consumption possibilities is preferred. (6) Markets exist for all the commodities (both present and future). (7) The price of a commodity is the number of accounting units which have to be paid *now* for the (future) availability of one unit of that commodity. (8) Market equilibrium is defined as a state where the demand for each commodity (both present and future) equals its supply.

Debreu's theory proves that such an economy has at least one set of equilibrium prices, that an equilibrium relative to a price system is an optimum and *vice versa*.

Debreu himself acknowledges that the above economy shows striking similarities to Hicks' pure "Futures Economy" in *Value and Capital*.[6] Since every producer knows his future production possibilities and every consumer knows his future consumption possibilities and future resources, everyone can foretell precisely his demand and supply functions

[5] However in Chapter 7 he shows that the theory is applicable to the case where the consumption sets, the production sets and the resources of the economy depend on uncertain events.

[6] J. R. Hicks, *op. cit.*, p. 136.

of commodities available in future periods. The prices which will be realized in future periods will be exactly the same as those which are currently established in the futures market. Thus market prices (like the sun in "Faust"[7]) will follow a course through periods which is fixed up in advance by the equilibrium prices of the pure "Futures Economy". In the economy where each trader has such perfect foresight the dynamic state can be analyzed in the same way as the stationary state.

### 3. AN ALTERNATIVE INTERPRETATION: HICKS' TEMPORARY EQUILIBRIUM MODEL

It must be realized that the pure "Futures Economy" can have no claims to be a good approximation to reality since actual futures markets are organized only for a small number of goods, locations, and periods (not too distant in the future). Each trader can only guess the future prices of those commodities for which the organized futures markets do not exist, and his price expectations may or may not be right. Even if each trader correctly knows his own demand and supply functions of future commodities, he does not know which prices match market demands and supplies as a whole. Current prices which are determined in current markets relative to people's expectations of futures prices would coincide with the corresponding ones in the perfect-equilibrium-over-time price set if no one made any mistakes in price expectations; but the market equilibrium is no more than a temporary equilibrium if people's price expectations are not right.

Hicks formulated his temporary equilibrium model by introducing the concept of price expectations into the traditional theory of general equilibrium.[8] In his analysis, however, the demonstration of whether or not such a system has a solution is no more than the mere counting of equations and unknowns, which had satisfied most economists up to that time. In this section we attempt to make Hicks' theory of temporary equilibrium well-founded by giving a new interpretation to Debreu's theory.

The commodities which are available in the present period (or in period 0) are called present commodities, and those which will be available in the future are called future commodities. Each trader draws up his plan of supply and demand of both present and future commodities simultaneously, and that plan depends not only on current prices and the current interest rate, but also on his expectations about future

[7] J. R. Hicks, *op. cit.*, p. 132.
[8] J. R. Hicks, *op. cit.*, pp. 124-25.

prices and future interest rates. We suppose, for simplicity, that no futures market exists, whereas markets for all present commodities exist. Let us suppose, further, that only short lending is allowed, or that all loans are made for the minimum period—one period. In such a spot economy with short lending the following budget equations will hold for any consumer:

$$\begin{aligned}&\text{Holdings of securities at the end of the previous period}\\&\quad+\text{Interest earned on them}+\text{Non-interest income of period 0}\\(1.0)\quad&=\text{Expenditure on commodities during period 0}\\&\quad+\text{Holdings of securities at the end of period 0,}\end{aligned}$$

$$\begin{aligned}&\text{Holdings of securities at the end of period } (t-1) \text{ of the future}\\&\quad+\text{Interest earned on them}+\text{Non-interest income of period } t\\(1.t)\quad&=\text{Expenditure on commodities during period } t\\&\quad+\text{Holdings of securities at the end of period } t,\\&\qquad\qquad t=1,2,\cdots,T,\end{aligned}$$

where $T$ is the last period of the time horizon over which utility is maximized. Similarly the following budget equations will hold for any producer:

$$\begin{aligned}&\text{Holdings of securities at the end of the previous period}\\&\quad+\text{Interest earned on them}+\text{Value of output during period 0}\\(2.0)\quad&=\text{Value of input during period 0}\\&\quad+\text{Holdings of securities at the end of period 0}\\&\quad+\text{Dividends in period 0,}\end{aligned}$$

$$\begin{aligned}&\text{Holdings of securities at the end of period } (t-1) \text{ of the future}\\&\quad+\text{Interest earned on them}+\text{Value of output during period } t\\(2.t)\quad&=\text{Value of input during period } t\\&\quad+\text{Holdings of securities at the end of period } t\\&\quad+\text{Dividends in period } t,\qquad t=1,2,\cdots,T,\end{aligned}$$

where each producer's time horizon is assumed to be equal to that of each consumer. Holdings of securities represent lending if they are positive, and borrowing if they are negative.

Debreu's fundamental theorem of the existence of competitive equilibrium is, in its simplest form, stated as follows:[9]

THEOREM. *Suppose that* $x_1, x_2, \cdots, x_n$ *are continuous functions of* $p_1, p_2, \cdots, p_n$, *and that*

$$\sum_{i=1}^{n} p_i x_i = 0$$

[9] G. Debreu, *op. cit.*, p. 82. This simplest form was independently noticed by Professors D. Gale and H. Nikaidô

*is an identity in variables* $p_1, p_2, \cdots, p_n$ *(Walras' law). Then there exist* $p_i \geqq 0$, $\sum_{i=1}^{n} p_i > 0$ $(i = 1, 2, \cdots, n)$ *such that* $x_i \leqq 0$ $(i = 1, 2, \cdots, n)$.

With the intention of applying this theorem to the existence problem that confronts us, we deduce the following two types of Walras' law. First, by summing (1.0) and (2.0) over all consumers and producers, we obtain *Walras' law within a period*

$$\begin{aligned}(3)\quad &(\text{Current expenditure by consumers} - \text{Value of current output})\\ &+ (\text{Current dividends} + \text{Value of current input}\\ &\qquad - \text{Current non-interest income of consumers})\\ &+ (\text{Current lending} - \text{Current borrowing}) = 0\ .^{10}\end{aligned}$$

Second, we obtain *Walras' law over time* by discounting $(1.t)$ and $(2.t)$ $(t = 0, 1, \cdots, T)$ by appropriate discount factors, corresponding to future rates of interest, and summing them over all consumers and producers.

$$\begin{aligned}(4)\quad &(\text{Current expenditure by consumers} - \text{Value of current output})\\ &+ (\text{Current dividends} + \text{Value of current input}\\ &\qquad - \text{Current non-interest income of consumers})\\ &+ (\text{Present value of future expenditure by consumers}\\ &\qquad - \text{Present value of future output})\\ &+ (\text{Present value of future dividends}\\ &\qquad + \text{Present value of future input}\\ &\quad - \text{Present value of future non-interest income of consumers})\\ &+ (\text{Present value of securities held at the end of period } T)\\ &= 0.\end{aligned}$$

If we assume, as Debreu does, that no one wants to hold securities at the end of period $T$, the last parenthesized part of (4) is identically zero. Then, from (3) and (4),

$$\begin{aligned}(5)\quad &(\text{Current lending} - \text{Current borrowing})\\ &= (\text{Present value of future expenditure by consumers}\\ &\qquad - \text{Present value of future output})\\ &+ (\text{Present value of future dividends}\\ &\qquad + \text{Present value of future input}\\ &\quad - \text{Present value of future non-interest income of consumers}).\end{aligned}$$

The right-hand side can be rewritten as

[10] The total sum of securities held at the end of the previous period vanishes because *ex post* lending must equal *ex post* borrowing.

(Discount factor in period 0)
× [(Discounted value of future expenditure at the beginning of period 1
− Discounted value of future output at the beginning of period 1)
+ (Discounted value of future dividends at the beginning of period 1
+ Discounted value of future input at the beginning of period 1
− Discounted value of future non-interest income of consumers at the beginning of period 1)].

The part within square brackets may be regarded as a continuous function of all current prices and current rate of interest, as well as of all expected future prices and expected future rates of interest. If we assume that expected future prices and expected future rates of interest are functions of current prices and the current discount factor, the part within square brackets (*i.e.*, current excess demand for securities) becomes a function of current prices and the current discount factor. In addition, the first two parenthesized parts of (3) are the sum of the products of current prices and excess demands for current commodities which are continuous functions of current prices and the current discount factor. Hence, by applying the fundamental theorem to Walras' law (3), we can prove the existence of temporary equilibrium; *i. e.*, there exist a set of current prices and a discount factor which make the excess demands for all commodities (including securities) non-positive.[11]

It should be noted that Debreu uses *Walras' law over time* (4) in order to prove the existence of equilibrium over time, with which he is concerned.

## 4. TOWARDS A GENERAL THEORY

In the temporary equilibrium system we have a market for securities, instead of markets for future commodities in the pure "Futures Economy". The reason why the excess demand function of securities has been derived from those of future commodities is that we have made the following two assumptions: (1) There exists no money. (2) No one wants to hold securities at the end of period $T$. If either assumption does not hold, we find no specific relation between the excess demand function of securities and the excess demand functions of future commodities.

[11] As there are no markets for future commodities, the individual excess demands for them, which appear on the right-hand side of (5), may not be non-positive, but the whole part of the right-hand side becomes non-positive from the condition for equilibrium in the securities market.

Let us first be concerned with an economy in which consumers and producers can hold their assets in the form of money; we assume, for simplicity, that they keep holdings of cash balances unchanged from the end of period 0 to the end of period $T$. In such a situation, (5) is replaced by

(Demand for cash balances − Existing quantity of money)
+ (Lending in period 0 − Borrowing in period 0)
= The right-hand side of (5).

Hence, we cannot derive the excess demand function of securities from those of future commodities alone. In order to have it well-defined we must analyze the *intratemporal* liquidity preference between money and securities.

Secondly, when producers and consumers wish to hold liquid assets in the form of securities at the end of period $T$, the following equation will take the place of (5):

(Lending in period 0 − Borrowing in period 0)
= The right-hand side of (5)
+ Present value of securities held at the end of period $T$.

It is evident that the excess demand for securities in period 0 depends not only on excess demands for future commodities, but also on the *intertemporal* liquidity preference between securities held at the end of period 0 and those held at the end of period $T$.

The above arguments invite us to analyze consumers' and producers' liquidity preferences. This will be achieved by inserting a new chapter "Bankers", written in set-theoretical terms, between "Consumers" and "Equilibrium" in Debreu's *Theory of Value*.[12]

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[12] He explicitly recognizes that he has offered no theory of money. The integration of money in the Walrasian general equilibrium theory has been done by Professor Patinkin, although his theory is based on the traditional crude assumption that equality between the number of equations and unknowns in a system is sufficient for the existence of a solution. See his *Money, Interest, and Prices* (Evanston: Row Peterson and Co., 1953).

## ERRATA

Benoit Mandelbrot writes regarding his recent article, "The Pareto-Lévy Law and the Distribution of Income," Vol. 1 (May, 1960):

The following slips have been pointed out by R. E. Levitan. I should like to take this opportunity to make a few ambiguous sentences more explicit. (Lines counted from the bottom of the page are designated by negative numbers.)

Page 84, line −15: After $kv''$, add "(which *a priori* is a function of $k$ and of $v(t)$)".

Page 84, line 15: Replace $+ M$ by $- M$.

Page 87, line 19: Read $P(u) \sim u^{-\alpha}(u^*)^{\alpha}[\Gamma(1-\alpha)]^{-1}$.

Page 87, line −10: Read $u^0 = u^*[\Gamma(1-\alpha)]^{-1/\alpha}$.

Page 89, line −6: Replace "are differences of" by "are either sums (obviously) or differences of".

Page 89, line −1: Replace $-iM$ by $iM$ and $\cos(\alpha\pi/2)$ by $|\cos(\alpha\pi/2)|$.

Page 90, line 4: Make the same corrections as on page 89, line −1.

Page 90, line −6 of footnote: The expression "we find" means that this result can be shown to hold. But the argument is *not* given in this footnote. The complete proof is long and can be found in [8].

Page 92, line −13: Before the beginning of the line add: "weak Pareto".

Page 92, line −1: Delete "by integration by parts".

Page 93: Delete the last integrals of each of the two formulas near the top of the page, since these formulas ceased to be valid when we changed the domain of integration from $(0, \infty)$ to $(-\infty, \infty)$. Place "minus" in front of $\int_{-\infty}^{\infty} u^2 dP(u)$ in the second formula.

Page 103, line −4: Read $\alpha(u - u'')^{-\alpha-1}(u^0)^{\alpha}$.

SPRINGER VERLAG IN WIEN

# Der praktische Einsatz elektronischer Rechenautomaten

**Einführung in die Programmierung und den betriebswirtschaftlichen Einsatz elektronischer Rechenautomaten**

Von

**Ernst P. Billeter**

Leiter des Institutes für Automation und Professor für Statistik und Automation an der Universität Freiburg (Schweiz)

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